C^*-ALGEBRAS GENERALIZING BOTH RELATIVE CUNTZ-PIMSNER AND DOPLICHER-ROBERTS ALGEBRAS

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Abstract. We introduce and analyse the structure of C^*-algebras arising from ideals in right tensor C^*-precategories, which naturally generalize both relative Cuntz-Pimsner and Doplicher-Roberts algebras. We establish an explicit intrinsic construction of the algebras considered and prove a number of key results such as structure theorem, gauge-invariant uniqueness theorem or description of the gauge-invariant ideal structure. In particular, these statements give a new insight into the corresponding results for relative Cuntz-Pimsner algebras and are applied to Doplicher-Roberts algebras associated with C^*-correspondences.

Contents

Introduction

C^*-categories, the categorical analogues of (unital) C^*-algebras, arise quite naturally in different problems of representation theory, harmonic analysis, or cohomology theory, cf. [14], [32] and sources cited there. The recent interest in C^*-categories however is essentially due to a series of papers by S. Doplicher and J. E. Roberts where, motivated by questions arising in quantum field theory, they developed an abstract duality for compact groups, cf. [12]. In their scenario an object of the dual group is represented by a certain tensor C^*-category $\mathcal{T}$, and a machinery performing this duality rests on a construction of a C^*-algebra $\mathcal{O}_\rho$, influenced by Cuntz algebras [5], associated functorially to each object ρ of $\mathcal{T}$. This association of $\mathcal{O}_\rho$ can be applied, with no substantial modifications, to the case where $\mathcal{T}$ is just a *right tensor*

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C^*-category, i.e. a C^*-category for which the set of objects is a unital semigroup, with identity ι, and for any object $\tau \in \mathcal{T}$ there is a *-functor $\otimes 1_\tau : \mathcal{T} \to \mathcal{T}$ (which intuitively should be thought of as a tensoring on the right with identity 1_τ in the space of morphisms $\mathcal{T}(\tau,\tau)$) such that

$$\otimes 1_\tau : \mathcal{T}(\rho,\sigma) \to \mathcal{T}(\rho\tau,\sigma\tau), \quad \text{and} \quad \otimes 1_\iota = id, \qquad ((a\otimes 1_\tau)\otimes 1_\omega) = a \otimes 1_{\tau\omega},$$

where $\omega,\rho,\sigma \in \mathcal{T}$, $a \in \mathcal{T}(\rho,\sigma)$, and we write $a\otimes 1_\tau$ for an element customary denoted by $\otimes 1_\tau(a)$.

The construction of the single algebra $\mathcal{O}_\rho$, with ρ fixed, relies only on the semigroup $\{\rho^n\}_{n\in\mathbb{N}}$ generated by ρ (we adopt the convention that $0 \in \mathbb{N}$). In such a situation we may make our framework even more transparent by assuming that $\mathcal{T}$ is simply a C^*-category with $\mathbb{N}$ as the set of objects, equipped with a *-functor $\otimes 1 : \mathcal{T} \to \mathcal{T}$ such that $\otimes 1 : \mathcal{T}(n,m) \to \mathcal{T}(n+1,m+1)$, $n,m\in\mathbb{N}$. Then it makes sense to denote the *Doplicher-Roberts algebra* (associated to the object $1\in\mathbb{N}$) by $\mathcal{DR}(\mathcal{T})$. Such algebras are sometimes also called DR-algebras [32].

A somewhat different but closely related and very important class of algebras form the C^*-algebras associated with C^*-correspondences. This study was initiated by Pimsner [30]. More specifically, a C^*-correspondence X over a C^*-algebra A (sometimes called a Hilbert bimodule) is a right Hilbert A-module equipped with a left action ϕ of A by adjointable operators, and *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is constructed as a quotient of the *Toeplitz algebra* of X generated by the *Fock representation* of X on the Fock module $\mathcal{F}(X) = \bigotimes_{n=0}^\infty X^{\otimes n}$. Algebras arising in this way are known to comprise various C^*-algebras found in the literature: crossed products by automorphisms, partial crossed products, crossed products by endomorphisms, C^*-algebras of graphs (in particular Cuntz-Krieger algebras), Exel-Laca algebras, C^*-algebras of topological quivers, and many more. It has to be noted that originally Pimsner in his analysis assumed that the *left action ϕ on X is injective.* However, this seemingly only technical assumption turned out to be crucial. In particular, the efforts to remove this restriction resulted in a variety of approaches [28], [1], [10], [11], [16]. We singled out two of these. Firstly, the C^*-algebra $\mathcal{O}_X$ introduced by Katsura [16] seems to be the most natural candidate for $\mathcal{O}_X$ in the general case. It is the smallest C^*-algebra among C^*-algebras generated by injective representations of X admitting gauge actions, cf. [18, Prop. 7.14]. Secondly, the so-called *relative Cuntz-Pimsner algebras* $\mathcal{O}(J,X)$ of Muhly and Solel [28] possess traits of being the most general, since by particular choices of an ideal J in A, one can cover all the aforementioned constructions. Moreover, $\mathcal{O}(J,X)$ arise quite naturally, when one tries to understand the ideal structure of Cuntz-Pimsner algebras [10], [28], and when dealing with certain concrete problems of description of C^*-algebras generated by irreversible dynamical systems [8], [22], [21].

The relationship between Cuntz-Pimsner algebras and Doplicher-Roberts algebras $\mathcal{DR}(X)$ associated with a C^*-correspondence X was investigated in [13], [10] where it was assumed, as in Pimsner's paper [30], that the *left action ϕ on X is injective.* It was noticed that $\mathcal{DR}(X)$ is closely related but tends to be larger than $\mathcal{O}_X$. Namely, there are natural embeddings $\mathcal{O}_X \subset \mathcal{DR}(X) \subset \mathcal{O}_X^{**}$ and the equality $\mathcal{O}_X = \mathcal{DR}(X)$ holds, for instance, if X is finite projective, cf. [13, Prop. 3.2], [10, Cor. 6.3]. In particular, if $\mathcal{O}_X = \mathcal{DR}(X)$ the Doplicher-Roberts construction makes the analysis of the Cuntz-Pimsner algebra $\mathcal{O}_X$ very accessible and proves to be very useful, cf. [15]. It should be stressed that when ϕ is not injective, however, the relation between $\mathcal{DR}(X)$ and any of the algebras $\mathcal{O}(J,X)$ is far more elusive and remains

practically untouched. In this article we develope a general approach to overcome these problems.

Our main motivation and observation is that even though the categorical language of Doplicher and Roberts does not exactly fit into the formalism describing Cuntz-Pimsner algebras, in many aspects it is more natural and being *properly adapted* it clarifies the relationship between all the above-named constructions as well as shed much more light on their structures. To support this point of view let us note that the algebras $\mathcal{O}(J,X)$, $\mathcal{DR}(X)$, $\mathcal{DR}(\mathcal{T})$ are respectively spanned by images of the spaces of "compact" operators, adjointable operators, and abstract morphisms (arrows):

$$\mathcal{K}(X^{\otimes n}, X^{\otimes m}), \qquad \mathcal{L}(X^{\otimes n}, X^{\otimes m}), \qquad \mathcal{T}(n,m), \qquad n,m \in \mathbb{N},$$

where the above building blocks are "glued together" to form the corresponding algebras in procedures based essentially on the "tensoring" – either the natural tensoring on X, or an abstract tensoring on $\mathcal{T}$. Namely, if A is unital, the family $\mathcal{T}_X := \{\mathcal{L}(X^{\otimes n}, X^{\otimes m})\}_{n,m\in\mathbb{N}}$, where $X^{\otimes 0} := A$, with the tensoring on the right by the identity operator on X form a natural right tensor C^*-category. By definition $\mathcal{DR}(X) := \mathcal{DR}(\mathcal{T}_X)$ and the general Doplicher-Roberts algebra $\mathcal{DR}(\mathcal{T})$ is a C^*-algebra endowed with an action of the unit circle for which the k-spectral subspace is an inductive limit of the inductive sequence

$$\mathcal{T}(r+k,r) \xrightarrow{\otimes 1} \mathcal{T}(r+k+1,r+1) \xrightarrow{\otimes 1} \mathcal{T}(r+k+2,r+2) \xrightarrow{\otimes 1} \dots \ .$$

In particular, we have natural homomorphisms $i_{(n,m)} : \mathcal{T}(n,m) \to \mathcal{DR}(\mathcal{T})$ that form a representation of the C^*-category $\mathcal{T}$, [14], and clearly this representation determines the structure of $\mathcal{DR}(\mathcal{T})$. Similarly, the Fock representation (or any other universal representation) of X give rise to homomorphisms $i_{(n,m)} : \mathcal{K}(X^{\otimes n}, X^{\otimes m}) \to \mathcal{O}(J,X)$ which posses the same properties as the aforesaid representation of $\mathcal{T}$. It is evident and almost symptomatic that any analysis of $\mathcal{O}(J,X)$ leads to an analysis of the family $\{i_{(n,m)}\}_{n,m\in\mathbb{N}}$ of such representations. This traces the fact that within our unified approach we may transfer a very rich and well developed representation theory of relative Cuntz-Pimsner algebras [28], [10], [17], [18], onto the ground of Doplicher and Roberts. In particular, we may "improve" the construction of $\mathcal{DR}(\mathcal{T})$ so that the universal representation of $\mathcal{T}$ in $\mathcal{DR}(\mathcal{T})$ is injective, even when the right tensoring is not. On the other hand, generalizing the inductive limit construction of $\mathcal{DR}(\mathcal{T})$, which makes its structure very accessible, allow us to clear up the description of ideal structure of $\mathcal{O}(J,X)$ obtained in [10], [17].

There are two plain but important new principles that shine through our development:

1) It is more natural to work with C^*-precategories, the categorical analogues of (not necessarily unital) C^*-algebras, rather than with C^*-categories.
2) Not only right tensor C^*-precategories but also their ideals naturally give rise to universal C^*-algebras.

To support 1) note that the unit-existence-requirement embedded into the notion of a category causes dispensable technicalities and lingual inconsequence like that an ideal in a C^*-category, [14, Def. 1.6], may not be a C^*-category. Moreover, dealing with a C^*-correspondence X over a non-unital C^*-algebra A leads to the following problem: If ϕ is not non-degenerate there is no obvious right tensoring on the C^*-category $\{\mathcal{L}(X^{\otimes n}, X^{\otimes m})\}_{n,m\in\mathbb{N}}$, as there is no such extension of $\phi : A \to \mathcal{L}(X)$ up to the multiplier algebra $M(A) = \mathcal{L}(A)$ of A. A natural solution to that problem

is that one should consider a smaller C^*-precategory $\mathcal{T}_X$ where $\mathcal{T}_X(0,0) := A$ (then $\mathcal{T}_X$ is a C^*-category iff A is unital).

An important remark on account of 2) is that the C^*-precategory $\mathcal{K}_X$ where $\mathcal{K}_X(n,m) := \mathcal{K}(X^{\otimes n}, X^{\otimes m})$, $n, m \in \mathbb{N}$, forms an ideal in the right tensor C^*-precategory $\mathcal{T}_X$, but as a rule it is not a right tensor C^*-precategory itself. Indeed, a typical situation is that the tensor product of a "compact" operator with the identity is no longer "compact", so $\mathcal{K}_X$ is not preserved under the tensoring $\otimes 1$. As we will show this turns out not to be an obstacle. This is due to the new feature of our construction – it applies well not only to right tensor C^*-precategories but also to their ideals.

The article is organized as follows. We begin in Section 1 with a review of basic facts and objects related to C^*-correspondences that will be of interest to us. In particular, we introduce examples of C^*-correspondences associated with partial morphisms and directed graphs which we will use throughout the paper to give a dynamical and combinatorial interpretation of the theory presented. In Section 2, slightly modifying and extending the terminology of [14], we establish the rudiments of the theory of C^*-precategories. A very useful statement here is Theorem 2.6 which characterizes ideals in C^*-precategories via ideals in C^*-algebras. When applied to the C^*-precategory $\mathcal{K}_X$ this gives a one-to-one correspondence $J \longleftrightarrow \mathcal{K}_X(J)$ between the ideals in A and $\mathcal{K}_X$, see Proposition 2.16.

One of the most important structures of our analysis – the right tensor C^*-precategories and their representation theory is undertaken in Section 3. We provide a definition of a *right tensor representation* as a representation of a C^*-precategory which is "compatible" with the tensoring, and what is very important this notion makes sense not only for right tensor C^*-precategories but also for their ideals. Theorem 3.12 says that such representations may be considered as generalizations of representations of C^*-correspondences. Following this line of thinking we give a new meaning to the notion of coisometricity introduced by Muhly and Solel [28], see Definitions 3.17, 3.19, and also Corollary 3.22.

In Section 4 we present three different definitions, and establish their equivalence, of the *C^*-algebra $\mathcal{O}_\mathcal{T}(\mathcal{K},\mathcal{J})$ of an ideal $\mathcal{K}$ in a right tensor C^*-precategory $\mathcal{T}$ relative to an ideal $\mathcal{J}$*. We define $\mathcal{O}_\mathcal{T}(\mathcal{K},\mathcal{J})$ as: a universal C^*-algebra with respect to right tensor representations of $\mathcal{K}$ coisometric on $\mathcal{J}$, Definition 4.3; an explicitly constructed algebra with explicit formulas for norm and algebraic operations, Subsection 4.2; and a C^*-algebra obtained via inductive limits formed from a specially constructed right tensor C^*-precategory $\mathcal{K}_\mathcal{J}$, page 32. Such a variety of points of view results in a numerous immediate interesting remarks. In particular, it allows us to reveal the relationships between the algebras $\mathcal{O}_\mathcal{T}(\mathcal{K},\mathcal{J})$, $\mathcal{DR}(\mathcal{T})$ and algebras admitting circle action, see Section 5.

The fundamental tool in our analysis of the ideal structure of $\mathcal{O}_\mathcal{T}(\mathcal{K},\mathcal{J})$ is *Structure Theorem* (Theorem 6.7) which generalizes the main goal of [10]. It states that the ideal $\mathcal{O}(\mathcal{N})$ in $\mathcal{O}_\mathcal{T}(\mathcal{K},\mathcal{J})$ generated by an invariant ideal $\mathcal{N}$ in $\mathcal{T}$ may be naturally identified as $\mathcal{O}_\mathcal{T}(\mathcal{K}\cap\mathcal{N},\mathcal{J}\cap\mathcal{N})$, and the quotient $\mathcal{O}_\mathcal{T}(\mathcal{K},\mathcal{J})/\mathcal{O}(\mathcal{N})$ identifies as $\mathcal{O}_{\mathcal{T}/\mathcal{N}}(\mathcal{K}/\mathcal{N},\mathcal{J}/\mathcal{N})$. Moreover, we show that the ideal $\mathcal{N}$ may be replaced by its $\mathcal{J}$-saturation $\mathcal{S}_\mathcal{J}(\mathcal{N})$ (a concept that generalizes X-saturation [29] and negative invariance [18]), so that our Structure Theorem actually gives an embedding of the lattice of invariant, $\mathcal{J}$-saturated ideals in $\mathcal{T}$ into the lattice of gauge-invariant

ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$. Another application of the Structure Theorem establishes procedures of reduction of relations defining $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$, Definition 6.8, Theorem 6.9. This broadens and deepens a topic started in [25], and is indispensable in our further considerations. In Subsection 6.1 we discuss how Theorem 6.7 improves [10, Thm 3.1] and why a gauge-invariant ideal $\mathcal{O}(I)$ in a relative Cuntz-Pimsner algebra $\mathcal{O}(J,X)$ is "merely" Morita equivalent to the corresponding relative Cuntz-Pimsner $\mathcal{O}(J\cap I, XI)$.

Our analogue of the *gauge-invariant uniqueness theorem* is Theorem 7.3. It extends the corresponding theorems for relative Cuntz-Pimsner algebras [10, Thm. 4.1], [29, Thm. 5.1], [17, Thm. 6.4], [18, Cor. 11.7]. The main novelty is the use of the right tensor C^*-precategory $\mathcal{K}_{\mathcal{J}}$ constructed in Theorem 4.12. In particular, we establish (in Theorem 7.6) a lattice isomorphism that characterizes the *gauge-invariant ideal structure* of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ in terms of invariant $\mathcal{K}_{\mathcal{J}}$-saturated ideals in $\mathcal{K}_{\mathcal{J}}$. Under certain additional assumptions, this result allows to obtain an analogous description in terms of invariant $\mathcal{J}$-saturated ideals in $\mathcal{K}$ (Theorem 7.8). The general relationship between invariant saturated ideals in $\mathcal{K}$ and $\mathcal{K}_{\mathcal{J}}$ (and thereby also gauge-invariant ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$) is complex; on the level of relative Cuntz-Pimsner algebras it is completely revealed in Theorem 7.16 where the role of T-pairs introduced in [18] is also clarified.

Aiming at a generalization of [10, Thm. 6.6], [13, Thm. 4.1] in Section 8 we examine the general conditions assuring that algebras of type $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ embeds into one another. In this direction we establish two useful results, Propositions 8.2, 8.5, which are non-trivial, inequivalent generalizations of [13, Prop. 3.2], [10, Cor. 6.3]. Motivated by these considerations we introduce an analogue of relative Cuntz-Pimsner algebras - *relative Doplicher-Roberts algebras* $\mathcal{D}R(J,X)$, Definition 8.7. In particular, we give necessary and sufficient conditions under which the natural embedding $\mathcal{O}(J_0,X)\subset \mathcal{D}R(J,X)$ holds.

In the final section, we describe representations of $\mathcal{D}R(J,X)$ that extends representations of $\mathcal{O}(J_0,X)$ (Proposition 9.3) and give criteria under which such a representation is faithful, see Theorem 9.4. Additionally we show that every faithful representation of $\mathcal{O}(J_0,X)$ extends to faithful representation of $\mathcal{D}R(J,X)$ for an appropriate J, see Theorem 9.6.

Following [18] we denote by $\mathbb{N}=\{0,1,2,\ldots\}$ the set of natural numbers, by $\mathbb{C}$ the set of complex numbers, and by S^1 the group of complex numbers with absolute value 1. We use a convention that $\gamma(A,B)=\{\gamma(a,b)\in D \mid a\in A, b\in B\}$ for a map $\gamma\colon A\times B\to D$ such as inner products, multiplications or representations. We denote by $\mathrm{span}\{\cdots\}$ a linear spans of $\{\cdots\}$, and by $\overline{\mathrm{span}}\{\cdots\}$ the closure of $\mathrm{span}\{\cdots\}$.

1. Preliminaries on C^*-correspondences

We adopt the standard notations and definitions of objects related to (right) Hilbert C^*-modules, cf. [26], [31]. In particular, we denote by X and Y Hilbert modules over a C^*-algebra A; $\mathcal{L}(X,Y)$ stands for the space of adjointable operators from X into Y; and $\mathcal{K}(X,Y)$ is the space of "compact" operators in $\mathcal{L}(X,Y)$, that is $\mathcal{K}(X,Y)=\overline{\mathrm{span}}\{\Theta_{y,x}: x\in X, y\in Y\}$ where $\Theta_{y,x}(z)=y\langle x,z\rangle_A$, $z\in X$. If I is an ideal in A (by which we always mean a closed two-sided ideal), then XI is both a Hilbert A-submodule of X and a Hilbert I-module, as we have

$$XI=\{xi: x\in X,\, i\in I\}=\{x\in X:\langle x,y\rangle_A\in I \text{ for all } y\in X\},$$

cf. [18, Prop. 1.3]. We now discuss the possibility of extension of a natural identification of $\mathcal{K}(XI)$ as subalgebra of $\mathcal{K}(X)$ used in [18], [10].

Lemma 1.1. *The equalities*

$$\begin{aligned}\overline{\operatorname{span}}\{\Theta_{y,x} : x \in X, y \in YI\} &= \overline{\operatorname{span}}\{\Theta_{y,x} : x \in XI, y \in Y\}\\ &= \overline{\operatorname{span}}\{\Theta_{y,x} : x \in XI, y \in YI\}\end{aligned}$$

establish natural identifications

$$\mathcal{K}(X, YI) = \mathcal{K}(XI, Y) = \mathcal{K}(XI, YI) \subset \mathcal{K}(X, Y).$$

Proof. Clear by definitions, cf. [26, (1.6)], [18]. ∎

The above identifications do not carry over onto the ground of adjointable maps where the problem becomes more subtle due to a possible lack of adjoint operators. The relevant problem is solved by the following lemma.

Lemma 1.2. *We have an inclusion $\mathcal{L}(XI, Y) \subset \mathcal{L}(XI, YI)$ and a natural embedding $\mathcal{L}(X, YI) \hookrightarrow \mathcal{L}(XI, YI)$ obtained by restriction $a \mapsto a|_{XI}$ of mappings. Moreover under this embedding we have*

$$\mathcal{L}(X, YI) \cap \mathcal{L}(X, Y) = \mathcal{L}(XI, Y) \cap \mathcal{L}(X, YI). \tag{1}$$

In particular the Hasse diagram of the partial ordering given via inclusions and embeddings induced by restriction is presented on Fig. 1.

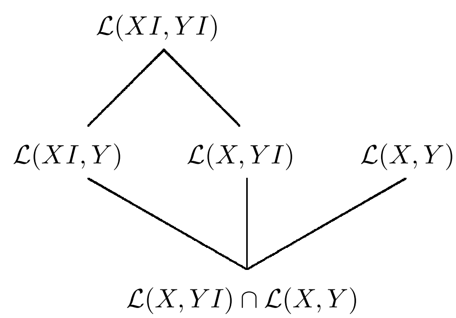

FIGURE 1. Hasse diagram for natural embeddings

Proof. If $a \in \mathcal{L}(XI, Y)$, then by A-linearity the range of a is contained in YI and $a \in \mathcal{L}(XI, YI)$ with the adjoint given by restriction of $a^* \in \mathcal{L}(Y, XI)$ to YI. Clearly, restriction $a \mapsto a|_{XI}$ defines a homomorphism from $\mathcal{L}(X, YI)$ to $\mathcal{L}(XI, YI)$. To see that it is injective denote by $\{\mu_\lambda\}_{\lambda\in\Lambda}$ an approximate unit for I, and note that if $a, b \in \mathcal{L}(X, YI)$ are such that $a|_{XI} = b|_{XI}$, then for $x \in X$ we have

$$ax = \lim_\lambda (ax)\mu_\lambda = \lim_\lambda (ax\mu_\lambda) = \lim_\lambda (bx\mu_\lambda) = \lim_\lambda (bx)\mu_\lambda = bx.$$

Hence we have the embedding $\mathcal{L}(X, YI) \subset \mathcal{L}(XI, YI)$, and using it one immediately gets

$$\mathcal{L}(XI, Y) \cap \mathcal{L}(X, YI) = \{a \in \mathcal{L}(XI, Y) : \exists_{\widetilde{a}\in\mathcal{L}(X,YI)} \widetilde{a}|_{XI} = a\}$$

and $\mathcal{L}(X, YI) \cap \mathcal{L}(X, Y) \subset \{a \in \mathcal{L}(XI, Y) : \exists_{\widetilde{a}\in\mathcal{L}(X,YI)} \widetilde{a}|_{XI} = a\}$. Thus to prove (1) it suffices to show that if $\widetilde{a} \in \mathcal{L}(X, YI)$ is such that $a := \widetilde{a}|_{XI} \in \mathcal{L}(XI, Y)$, then $\widetilde{a} \in \mathcal{L}(X, Y)$. But this follows, since $a^* \in \mathcal{L}(Y, XI)$ and thus for $x \in X$, $y \in Y$ we have

$$\langle \widetilde{a}x, y\rangle_A = \lim_\lambda \langle (\widetilde{a}x)\mu_\lambda, y\rangle_A = \lim_\lambda \langle a(x\mu_\lambda), y\rangle_A = \lim_\lambda \langle x\mu_\lambda, a^*y\rangle_A$$

$$= \lim_{\lambda} \langle x, (a^* y)\mu_\lambda \rangle_A = \langle x, a^* y \rangle_A,$$

that is a^* is adjoint to $\widetilde{a}$ treated as an operator from X to Y. ■

The above identifications become completely transparent when viewing X and Y as Hilbert modules in a C^*-algebra $\mathcal{M}$, cf. [13, Prop. 2.1]. In general, the spaces which are incomparable in the Hasse diagram on Fig. 1 are indeed incomparable (to see that one can adopt for instance [31, Ex. 2.19]). For our purposes, an important fact is

Corollary 1.3. *The C^*-algebra $\mathcal{K}(XI)$ is an ideal in $\mathcal{L}(X, XI) \cap \mathcal{L}(X)$ which in turn is an ideal in the C^*-algebra $\mathcal{L}(X)$.*

For an ideal I in a C^*-algebra A we may consider the quotient space X/XI as a Hilbert A/I-module with an A/I-valued inner product and right action of A/I given by

$$\langle q(x), q(y) \rangle_{A/I} := q(\langle x, y \rangle_A), \qquad q(x)q(a) = q(xa)$$

where q denotes both the quotient maps $A \to A/I$ and $X \to X/XI$, cf. [10, Lem. 2.1]. Moreover, we have a natural map $q : \mathcal{L}(X) \to \mathcal{L}(X/XI)$ where $q(a)q(x) = q(ax)$ for $a \in \mathcal{L}(X)$ and $x \in X$.

Lemma 1.4. *The kernel of the map $q : \mathcal{L}(X) \to \mathcal{L}(X/XI)$ is $\mathcal{L}(X, XI) \cap \mathcal{L}(X)$ and the restriction of q to $\mathcal{K}(X)$ is a surjection onto $\mathcal{K}(X/XI)$ whose kernel is $\mathcal{K}(XI)$.*

Proof. See [18, Lem. 1.6] and remarks preceding this statement. ■

Corollary 1.5 (Lem. 2.6 [10])**.** *We have a natural isomorphism $\mathcal{K}(X)/\mathcal{K}(XI) \cong \mathcal{K}(X/XI)$.*

We follow a convention which seems to become standard, and for Hilbert modules with left action we use the term C^*-correspondence, while we reserve the term Hilbert bimodule for an object with an additional structure (cf. Definition 1.9 and Proposition 1.10 below).

Definition 1.6. *A C^*-correspondence X over a C^*-algebra A* is a (right) Hilbert A-module equipped with a *-homomorphism $\phi : A \to \mathcal{L}(X)$. We refer to ϕ as the left action of the C^*-correspondence X and write $a \cdot x := \phi(a)x$.

Let us fix a C^*-correspondence X over a C^*-algebra A and a Hilbert A-module Y. There is a naturally defined tensor product Hilbert A-module $Y \otimes X$, cf. [26], [31] or [17]. An ideal I in A is called *X-invariant* if $\varphi(I)X \subset XI$, and for such an ideal the quotient A/I-module X/XI with right action $q(a)q(x) = q(\varphi(a)x)$ becomes a C^*-correspondence over A/I, cf. [10, Lemma 2.3], [18] [15]. In particular, we may consider two Hilbert A/I-modules $Y/YI \otimes X/XI$ and $(Y \otimes X)/(Y \otimes XI)$.

Lemma 1.7. *For an X-invariant ideal I in A we have a natural isomorphism of Hilbert modules*

$$(Y/YI) \otimes (X/XI) \cong (Y \otimes X)/(Y \otimes XI).$$

Proof. Let $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $i, j \in I$. Then

$$(y_1 + y_2 j) \otimes (x_1 + x_2 i) = y_1 \otimes x_1 + \Big(y_1 \otimes x_2 i + y_2 \otimes \varphi(j) x_1 + y_2 j \otimes x_2 i \Big)$$

where (by X-invariance of I) the term in brackets belongs to $Y \otimes XI$. This shows that the mapping

$$(y + YI) \otimes (x + XI) \longmapsto y \otimes x + Y \otimes XI$$

is well defined. Clearly, it is surjective and A/I-linear. The simple calculation:

$$\langle q(y_1)\otimes q(x_1), q(y_2)\otimes q(x_2)\rangle_{A/I} = \langle q(x_1), q(\varphi\langle y_1,y_2\rangle_A) q(x_2)\rangle_{A/I}$$

$$= q(\langle x_1, \varphi(\langle y_1,y_2\rangle_A)x_2\rangle_A) = q(\langle y_1\otimes x_1, y_2\otimes x_2\rangle_A) = \langle q(y_1\otimes x_1), q(y_2\otimes x_2)\rangle_{A/I},$$

shows that the above mapping preserves A/I-valued inner products and hence is isometric. ■

We have a natural homomorphism $\mathcal{L}(Y)\ni a\to a\otimes 1\in\mathcal{L}(Y\otimes X)$ where

$$(a\otimes 1)(y\otimes x) := ay\otimes x, \qquad x\in X,\ y\in Y. \tag{2}$$

The properties of this homomorphism are related to objects that will play important role throughout this paper. We define

$$J(X) := \varphi^{-1}(\mathcal{K}(X))$$

which is an ideal in A. If J is an ideal in A we define

$$J^\perp = \{a\in A : aJ = \{0\}\}$$

which is also an ideal in A called an *annihilator* of J. It is a unique ideal in A such that $J^\perp\cap J = \{0\}$ and for any ideal I in A we have $J\cap I = \{0\} \implies I\subset J^\perp$.

Lemma 1.8. (cf. [10, Lemma 4.2]). *Suppose that X is C^*-correspondence over A and Y is a right Hilbert A-module.*

i) *The map $a\mapsto a\otimes 1$ restricted to $\mathcal{L}(Y, Y(\ker\phi)^\perp)\cap\mathcal{L}(Y)$ is isometric.*
ii) *If $a\in\mathcal{L}(Y, Y(\ker\phi)^\perp)\cap\mathcal{L}(Y)$ and $a\otimes 1\in\mathcal{K}(Y\otimes X)$, then $a\in\mathcal{K}(Y)$.*
iii) *If $a\otimes 1\in\mathcal{K}(Y\otimes X)$ and $a\in\mathcal{K}(Y)$, then $a\in\mathcal{K}(YJ(X))$.*

Proof. i). Let $a\in\mathcal{L}(Y, Y(\ker\phi)^\perp)$. It suffices to check the inequality $\|a\|\le\|a\otimes 1\|$. For that purpose take an arbitrary $y\in Y$. Since $\langle ay, ay\rangle_A\in(\ker\phi)^\perp$ and ϕ is isometric on $(\ker\phi)^\perp$ we have $\|\phi(\langle ay,ay\rangle_A)\| = \|ay\|^2$. Since $\phi(\langle ay,ay\rangle_A)$ is positive, for each $\varepsilon>0$, there exists $x\in X$ such that $\|x\|=1$ and

$$\|\langle x, \phi(\langle ay,ay\rangle_A)x\rangle_A\| \ge \|\phi(\langle ay,ay\rangle_A)\| - \varepsilon = \|ay\|^2-\varepsilon.$$

Thus

$$\|(a\otimes 1)(y\otimes x)\|^2 = \|\langle x, \phi(\langle ay,ay\rangle_A)x\rangle_A\| \ge \|ay\|^2-\varepsilon$$

and as $\|(y\otimes x)\|\le\|y\|$ we get $\|a\otimes 1\|\ge\|a\|$.
ii). Let $\{\mu_\lambda\}_\lambda$ be an approximate unit for $\mathcal{K}(Y)$. Since $a\otimes 1\in\mathcal{K}(Y\otimes X)$, by item i) we get

$$0 = \lim_\lambda\|a\otimes 1 - (\mu_\lambda\otimes 1)(a\otimes 1)\| = \lim_\lambda\|a-\mu_\lambda a\|.$$

Hence a is "compact", cf. the proof of [10, Lemma 4.2 2)].
iii). See the second part of the proof of [10, Lemma 4.2 (2)]. ■

Following [17] we clarify the relationship between C^*-correspondences and Hilbert bimodules.

Definition 1.9. We say that X is a *Hilbert A-bimodule* if it is at the same time a Hilbert left A-module and a Hilbert right A-module with sesqui-linear forms ${}_A\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_A$ related via the so-called imprimitivity condition:

$$x\cdot\langle y,z\rangle_A = {}_A\langle x,y\rangle\cdot z, \qquad \text{for all } x,y,z\in X. \tag{3}$$

It follows from (3) that left action in a Hilbert bimodule acts by adjointable maps and hence every Hilbert C^*-bimodule is a C^*-correspondence, cf. [16, 3.3]. In the converse direction we have

Proposition 1.10. *Let X be a C^*-correspondence. The following conditions are equivalent:*

i) *X is a Hilbert A-bimodule (that is there exists a sesqui-linear form ${}_A\langle\cdot,\cdot\rangle$ for which X together with left action ϕ becomes a left Hilbert A-module satisfying* (3)*),*

ii) *there is a function ${}_A\langle\cdot,\cdot\rangle : X \times X \to (\ker\phi)^\perp$ such that*

$$\phi({}_A\langle x,y\rangle) = \Theta_{x,y}, \qquad x,y \in X, \tag{4}$$

iii) *the mapping $\phi : (\ker\phi)^\perp \cap J(X) \to \mathcal{K}(X)$ is onto (hence it is automatically an isomorphism).*

The objects in i) and ii) are determined uniquely, the function ${}_A\langle\cdot,\cdot\rangle$ from item ii) coincides with the inner product from item i) and

$${}_A\langle x,y\rangle = \phi^{-1}(\Theta_{x,y}), \qquad x,y \in X, \tag{5}$$

where ϕ^{-1} is the inverse to the isomorphism $\phi : (\ker\phi)^\perp \cap J(X) \to \mathcal{K}(X)$. Moreover denoting $\overline{{}_A\langle X,X\rangle} := \overline{\operatorname{span}}\{{}_A\langle x,y\rangle : x,y \in X\}$ we have

$$\overline{{}_A\langle X,X\rangle} = (\ker\phi)^\perp \cap J(X).$$

Proof. i)$\Rightarrow$ ii). One easily sees that conditions (4) and (3) are equivalent. Since ϕ is injective on $\overline{{}_A\langle X,X\rangle}$ and $\overline{{}_A\langle X,X\rangle}$ is an ideal we have $\overline{{}_A\langle X,X\rangle} \subset (\ker\phi)^\perp$ and thus ii) holds.
ii)$\Rightarrow$ iii). By (4) ϕ maps the set $\overline{{}_A\langle X,X\rangle}$ onto $\mathcal{K}(X)$. Since $\overline{{}_A\langle X,X\rangle} \subset (\ker\phi)^\perp \cap J(X)$ and ϕ is injective on $(\ker\phi)^\perp$ we see that $\phi : (\ker\phi)^\perp \cap J(X) \to \mathcal{K}(X)$ is an isomorphism. In particular, we have $\overline{{}_A\langle X,X\rangle} = (\ker\phi)^\perp \cap J(X)$ and (5) holds.
iii)$\Rightarrow$ i). It is straightforward to check that ${}_A\langle\cdot,\cdot\rangle$ defined by (5) is an inner product for the left A-module X. Indeed, since $\phi : (\ker\phi)^\perp \cap J(X) \to \mathcal{K}(X)$ is an isomorphism, it suffices to note that, for $x,y,z \in X$ and $a \in A$, we have

$$\Theta_{x+a\cdot y,z} = \Theta_{x,z} + a\Theta_{y,z}, \qquad \text{and} \qquad (\Theta_{x,y})^* = \Theta_{y,x}.$$

As we already noted, (4) and (3) are equivalent, and thus i) holds. ∎

Example 1.11 (C^*-correspondence of a partial morphism)**.** By a *partial morphism of a C^*-algebra A* we mean a *-homomorphism $\varphi : A \to M(A_0)$ from A to the multiplier algebra $M(A_0)$ of a hereditary subalgebra A_0 of A such that $\varphi(A)A_0 = A_0$, cf. [16]. We construct a C^*-correspondence X_φ from φ in the following way. We let $X_\varphi := A_0A$ and put

$$a \cdot x := \varphi(a)x, \quad x \cdot a := xa, \quad \text{and} \quad \langle x,y\rangle_A := x^*y,$$

where $a \in A$, $x,y \in X_\varphi$. For such a C^*-correspondence we have $J(X_\varphi) = \varphi^{-1}(A_0)$. To assert when X_φ is a Hilbert bimodule we slightly extend R. Exel's definition of a partial automorphism [9, Defn. 3.1] (which agrees with ours when A_0 is an ideal). By a *partial automorphism* of A we shall mean a triple (θ, I, A_0) consisting of an ideal I in A, a hereditary subalgebra A_0 of A and an isomorphism $\theta : I \to A_0$. A partial automorphism (θ, I, A_0) give rise to a partial morphism $\varphi : A \to M(A_0)$ via the formula $\varphi(a)b := \theta(a\theta^{-1}(b))$, $a \in A$, $b \in A_0$, and then we have $I = (\ker\varphi)^\perp \cap \varphi^{-1}(A_0)$. Conversely, if $\varphi : A \to M(A_0)$ is a partial morphism such that φ restricted to $I := (\ker\varphi)^\perp \cap \varphi^{-1}(A_0)$ is an isomorphism onto A_0, then φ arises from a partial automorphism (θ, I, A_0) where $\theta := \varphi|_I$.
Thus, in view of Proposition 1.10, a C^*-correspondence X_φ is a Hilbert bimodule

if and only if φ arises from a partial automorphism θ, and then the "left" inner product is given by

$$_A\langle x, y\rangle := \theta^{-1}(xy^*).$$

We indicate that every endomorphism $\delta : A \to A$ of a C^*-algebra A may be treated as a partial morphism where $A_0 = \delta(A)A\delta(A)$. In this sense a partial morphism φ is an endomorphism iff $\varphi^{-1}(A_0) = A$. We shall denote a C^*-correspondence arising from endomorphism α by X_α. It was shown in [20, Prop. 1.9] that if A is unital, then $X_\alpha = \alpha(1)\mathcal{A}$ is a Hilbert bimodule iff there exists a *complete transfer operator* for α introduced in [4] - it is a bounded, positive linear map $\mathcal{L} : A \to A$ such that

$$\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b), \qquad \text{and} \qquad \alpha(\mathcal{L}(1)) = \alpha(1), \qquad a, b \in \mathcal{A},$$

($\mathcal{L}$ exists iff the kernel of α is unital and the range is hereditary, see [20]). If this is the case the "left" inner product is given by

$$_A\langle x, y\rangle := \mathcal{L}(xy^*).$$

Example 1.12 (C^*-correspondence of a directed graph)**.** Suppose $E = (E^0, E^1, r, s)$ is a *directed graph* with vertex set E^0, edge set E^1, and $r, s : E^1 \to E^0$ describing the range and the source of edges. A *C^*-correspondence X_E of the graph E* is defined in the following manner, cf. [11], [10], [29]. The space X_E consists of functions $x : E^1 \to \mathbb{C}$ for which

$$v \in E^0 \longmapsto \sum_{\{e \in E^1 : r(e) = v\}} |x(e)|^2$$

belongs to $A := C_0(E^0)$, and X_E is a C^*-correspondence over A with the operations

$$\begin{aligned}
(x \cdot a)(e) &:= x(e)a(r(e)) \ \text{ for } e \in E^1, \\
\langle x, y\rangle_A(v) &:= \sum_{\{e \in E^1 : r(e) = v\}} \overline{x(e)}y(e) \ \text{ for } v \in E^0, \text{ and} \\
(a \cdot x)(e) &:= a(s(e))x(e) \ \text{ for } e \in E^1.
\end{aligned}$$

We note that X_E and A are respectively spanned by the point masses $\{\delta_f : f \in E^1\}$ and $\{\delta_v : v \in E^0\}$. In particular

$$(\ker \phi)^\perp = \overline{\operatorname{span}}\{\delta_v : 0 < |s^{-1}(v)|\}, \qquad J(X_E) = \overline{\operatorname{span}}\{\delta_v : |s^{-1}(v)| < +\infty\},$$

cf. for instance [29]. Moreover, if $v \in E^0$ emits finitely many edges and $f, g \in E^1$,

$$\phi(\delta_v) = \sum_{\{e \in E^1 : s(e) = v\}} \Theta_{\delta_e, \delta_e}, \qquad \text{and} \qquad \Theta_{\delta_f, \delta_g} \neq 0 \iff r(f) = r(g).$$

It follows that X_E is a Hilbert bimodule iff every vertex of the graph E emits and receives at most one edge (equivalently maps r, s are injective). If this is the case the left and right inner products are zero on the complement of $r(E^1)$ and $s(E^1)$, respectively, and

$$\langle x, y\rangle_A(r(e)) = \overline{x(e)}y(e), \qquad {}_A\langle x, y\rangle(s(e)) = x(e)\overline{y(e)}.$$

In particular, bimodule X_E may treated as a bimodule X_φ arising from a partial morphism $\varphi : C_0(E^0) \to C_b(r(E^1))$ defined by a homeomorphism $s \circ r^{-1} : r(E^1) \to s(E^1)$.

2. C^*-PRECATEGORIES AND THEIR IDEALS

For a notion of a precategory we adopt the standard definition of a category with the only exception that we drop the assumption of existence of identity morphisms. In the present paper we shall be interested in precategories with the class of objects being the set of natural numbers $\mathbb{N}$. However, for future reference and to underscore the categorical language, in this is section (and only in this section) we shall deal with general precategories.

Definition 2.1. A *precategory* $\mathcal{T}$ consists of a class of objects, denoted here by σ, ρ, τ, etc.; a class $\{\mathcal{T}(\sigma,\rho)\}_{\sigma,\rho\in\mathcal{T}}$ of disjoint classes of morphisms (arrows) indexed by ordered pairs of objects in $\mathcal{T}$; and a composition of morphisms $\mathcal{T}(\sigma,\rho)\times\mathcal{T}(\tau,\sigma) \ni (a,b) \to ab \in \mathcal{T}(\tau,\rho)$, which is associative, in the sense that $(ab)c = a(bc)$ whenever the compositions of morphisms a, b, c are allowable.

Clearly, we can equip (if necessary) every set of morphisms $\mathcal{T}(\sigma,\sigma)$ with an identity morphism in such a way that a given precategory $\mathcal{T}$ becomes a category $\mathcal{T}^+$. We generalize the notion of a C^*-category, cf. [14], [12], in an obvious fashion.

Definition 2.2. A precategory $\mathcal{T}$ is a *C^*-precategory* if each set of morphisms $\mathcal{T}(\sigma,\rho)$ is a complex Banach space, the composition of morphisms gives us a bilinear map

$$\mathcal{T}(\sigma,\rho)\times\mathcal{T}(\tau,\sigma) \ni (a,b) \to ab \in \mathcal{T}(\tau,\rho)$$

with $\|ab\| \leq \|a\|\cdot\|b\|$, and there is an antilinear involutive contravariant functor $^* : \mathcal{T} \to \mathcal{T}$ such that if $a \in \mathcal{T}(\sigma,\rho)$, then $a^* \in \mathcal{T}(\rho,\sigma)$ and the C^*-equality $\|a^*a\| = \|a\|^2$ holds. A C^*-precategory $\mathcal{T}$ where $\mathcal{T}$ is a category is a *C^*-category*.

Notational conventions 2.3. If $\mathcal{S}$ is a *sub-C^*-precategory* of $\mathcal{T}$, that is if $\mathcal{S}$ and $\mathcal{T}$ are two C^*-precategories such that each space $\mathcal{S}(\rho,\sigma)$ is a closed subspace of $\mathcal{T}(\rho,\sigma)$, we shall briefly write $\mathcal{S}\subset\mathcal{T}$. If $\mathcal{S}$ and $\mathcal{T}$ are two sub-C^*-precategories of another C^*-precategory, we denote by $\mathcal{T}\cap\mathcal{S}$ the C^*-precategory where $(\mathcal{S}\cap\mathcal{T})(\rho,\sigma) := \mathcal{S}(\rho,\sigma)\cap\mathcal{T}(\rho,\sigma)$.

Each space of morphisms $\mathcal{T}(\sigma,\sigma)$ in a C^*-precategory $\mathcal{T}$ is a C^*-algebra, and by the C^*-equality the functor "$*$" is isometric on every space $\mathcal{T}(\sigma,\rho)$. C^*-precategory $\mathcal{T}$ is a C^*-category if and only if every C^*-algebra $\mathcal{T}(\sigma,\sigma)$, $\sigma\in\mathcal{T}$, is unital. In general, by adjoining units to C^*-algebras $\mathcal{T}(\sigma,\sigma)$, $\sigma\in\mathcal{T}$, one may obtain a "unitization" of $\mathcal{T}$, that is a C^*-category which contains $\mathcal{T}$ as an ideal in the sense of the following definition, cf. [14, Def. 1.6].

Definition 2.4. By a (closed two-sided) *ideal $\mathcal{K}$ in a C^*-precategory $\mathcal{T}$* we shall mean a collection of Banach subspaces $\mathcal{K}(\rho,\sigma)$ in $\mathcal{T}(\rho,\sigma)$, $\sigma,\rho\in\mathcal{T}$, such that

$$a\,\mathcal{K}(\tau,\sigma)\subset\mathcal{K}(\tau,\rho) \quad \text{and} \quad \mathcal{K}(\tau,\rho)a\subset\mathcal{K}(\tau,\sigma), \quad \text{for any } a\in\mathcal{T}(\sigma,\rho),\ \sigma,\rho,\tau\in\mathcal{T}$$

Arguing as for C^*-algebras, cf. [14, Prop. 1.7], one may see that an ideal $\mathcal{K}$ in a C^*-precategory $\mathcal{T}$ is "self-adjoint", that is it is a C^*-precategory. Obviously, each space $\mathcal{K}(\sigma,\sigma)$ is a closed two-sided ideal in the C^*-algebra $\mathcal{T}(\sigma,\sigma)$. An interesting fact is that $\mathcal{K}$ is uniquely determined by these "diagonal ideals". In order to prove this, we apply the following simple lemma.

Lemma 2.5. *Let $\mathcal{T}$ be a C^*-precategory and let $a\in\mathcal{T}(\sigma,\rho)$. Let $\mathcal{K}(\sigma,\sigma)$, $\mathcal{K}(\rho,\rho)$ be arbitrary ideals in $\mathcal{T}(\sigma,\sigma)$ and $\mathcal{T}(\rho,\rho)$ respectively, and let $\{\mu_\lambda\}_\lambda$, $\{\nu_\lambda\}_\lambda$ be approximate units in $\mathcal{K}(\sigma,\sigma)$ and $\mathcal{K}(\rho,\rho)$ respectively. Then*

$$a^*a\in\mathcal{K}(\sigma,\sigma) \iff \lim_\lambda a\mu_\lambda = a,$$

$$aa^* \in \mathcal{K}(\rho,\rho) \iff \lim_\lambda \nu_\lambda a = a.$$

Proof. If $\lim_\lambda a\mu_\lambda = a$, then $\lim_\lambda a^*a\mu_\lambda = a^*a$ is in $\mathcal{K}(\sigma,\sigma)$. Conversely, if $a^*a \in \mathcal{K}(\sigma,\sigma)$ then

$$\|a - a\mu_\lambda\|^2 = \|(a-a\mu_\lambda)^*(a-a\mu_\lambda)\| \leq \|a^*a - a^*a\mu_\lambda\| + \|\mu_\lambda\|\cdot\|a^*a - a^*a\mu_\lambda\|,$$

what implies that $\mu_\lambda a$ converges to a. The second equivalence can be proved analogously. ■

Theorem 2.6 (Characterization of ideals in C^*-precategories)**.** *If $\mathcal{K}$ is an ideal in a C^*-precategory $\mathcal{T}$, then for every objects σ and ρ*

$$\mathcal{K}(\sigma,\rho) = \{a \in \mathcal{T}(\sigma,\rho) : aa^* \in \mathcal{K}(\rho,\rho)\} = \{a \in \mathcal{T}(\sigma,\rho) : a^*a \in \mathcal{K}(\sigma,\sigma)\}. \tag{6}$$

Conversely, if $\{\mathcal{K}(\sigma,\sigma)\}_{\sigma\in\mathcal{T}}$ is a class such that $\mathcal{K}(\sigma,\sigma)$ is an ideal in the C^*-algebra $\mathcal{T}(\sigma,\sigma)$, and the equality

$$\{a \in \mathcal{T}(\sigma,\rho) : aa^* \in \mathcal{K}(\rho,\rho)\} = \{a \in \mathcal{T}(\sigma,\rho) : a^*a \in \mathcal{K}(\sigma,\sigma)\} \tag{7}$$

holds for every $\sigma,\rho \in \mathcal{T}$, then relations (6) *define an ideal $\mathcal{K}$ in $\mathcal{T}$.*

Proof. Let $\mathcal{K}$ be an ideal in $\mathcal{T}$ and let $a \in \mathcal{T}(\sigma,\rho)$. If $a \in \mathcal{K}(\sigma,\rho)$ then $a^*a \in \mathcal{K}(\sigma,\sigma)$. Conversely, if $a^*a \in \mathcal{K}(\sigma,\sigma)$, then using Lemma 2.5 one gets $a \in \mathcal{K}(\sigma,\rho)$. Thus we see that $\mathcal{K}(\sigma,\rho) = \{a \in \mathcal{T}(\sigma,\rho) : aa^* \in \mathcal{K}(\rho,\rho)\}$. The equality $\mathcal{K}(\sigma,\rho) = \{a \in \mathcal{T}(\sigma,\rho) : a^*a \in \mathcal{K}(\sigma,\sigma)\}$ can be proved analogously.
To prove the second part of proposition fix a class of ideals $\{\mathcal{K}(\sigma,\sigma)\}_{\sigma\in\mathcal{T}}$ such that (7) holds and use (6) to define $\mathcal{K} = \{\mathcal{K}(\sigma,\rho)\}_{\sigma,\rho\in\mathcal{T}}$. If $a \in \mathcal{K}(\sigma,\rho)$, then $a^*a \in \mathcal{K}(\sigma,\sigma)$ and for arbitrary $b \in \mathcal{T}(\tau,\sigma)$ we have $(ab)^*(ab) = b^*(a^*a)b \in \mathcal{K}(\sigma,\sigma)$, what shows that $a\mathcal{T}(\tau,\sigma) \subset \mathcal{K}(\tau,\rho)$. From (6) it follows that the star functor preserves $\mathcal{K}$ and thus $\mathcal{K}(\tau,\rho)a = (a^*\mathcal{K}(\rho,\tau))^* \subset (\mathcal{K}(\sigma,\tau))^* = \mathcal{K}(\tau,\sigma)$ for $a \in \mathcal{T}(\sigma,\rho)$. ■

As an application of the above statement we construct an annihilator of an ideal in a C^*-precategory.

Proposition 2.7. *If $\mathcal{J}$ is an ideal in C^*-precategory $\mathcal{T}$, then there exists a unique ideal $\mathcal{J}^\perp$ in $\mathcal{T}$ such that*

$$\mathcal{J}^\perp(\sigma,\sigma) = \mathcal{J}(\sigma,\sigma)^\perp, \qquad \sigma \in \mathcal{T}.$$

Moreover, we have $\mathcal{J}^\perp \cap \mathcal{J} = \{0\}$ and for any ideal $\mathcal{K}$ in $\mathcal{T}$

$$\mathcal{J} \cap \mathcal{K} = \{0\} \implies \mathcal{K} \subset \mathcal{J}^\perp$$

where $\{0\}$ denotes the ideal consisting of zero morphisms.

The ideal $\mathcal{J}^\perp$ will be called *an annihilator of $\mathcal{J}$*.
Proof. To show the existence of $\mathcal{J}$, by Theorem 2.6, it suffices to verify that the ideals $\mathcal{J}(\sigma,\sigma)^\perp$, $\sigma \in \mathcal{T}$, satisfy condition (7). Assume *ad absurdum* that there exists $a \in \mathcal{J}(\sigma,\rho)$ such that $a^*a \notin \mathcal{J}(\sigma,\sigma)^\perp$ and $aa^* \in \mathcal{J}(\rho,\rho)^\perp$. Then there exists $b \in \mathcal{J}(\sigma,\sigma)$ such that $a^*ab \neq 0$ and we have the following sequence of implications

$$a^*ab \neq 0 \Longrightarrow ab \neq 0 \Longrightarrow abb^*a^* \neq 0 \Longrightarrow (abb^*a^*)(aa^*) \neq 0.$$

(to see that one may, for instance, represent morphisms of $\mathcal{J}$ as operators in a Hilbert space, see Remark 2.12). Consequently, $(abb^*a^*)(aa^*)$ is a non-zero element of $\mathcal{J}(\rho,\rho) \cap \mathcal{J}(\rho,\rho)^\perp$ which is an absurd. The remaining part of proposition is straightforward. ■

We now turn to discussion of structure-preserving maps between C^*-precategories.

Definition 2.8. A *homomorphism* Φ from a C^*-precategory $\mathcal{T}$ to a C^*-precategory $\mathcal{S}$ consists of a mapping $\mathcal{T} \ni \sigma \mapsto \Phi(\sigma) \in \mathcal{S}$ and linear operators $\mathcal{T}(\sigma,\rho) \ni a \mapsto \Phi(a) \in \mathcal{S}(\Phi(\sigma),\Phi(\rho))$, $\sigma,\rho \in \mathcal{T}$, such that

$$\Phi(a)\Phi(b) = \Phi(ab), \qquad \Phi(a^*) = \Phi(a)^*, \qquad a \in \mathcal{T}(\sigma,\rho),\ b \in \mathcal{T}(\tau,\sigma),\ \sigma,\rho,\tau \in \mathcal{T}.$$

The notions such as an *isomorphism*, *endomorphism*, etc., for C^*-precategories are defined in an obvious way.

In the case C^*-precatogories $\mathcal{T}$, $\mathcal{S}$ are C^*-categories a homomorphism $\Phi : \mathcal{T} \to \mathcal{S}$ is a functor iff it is "unital", that is for every $\sigma \in \mathcal{T}$, Φ maps the unit in $\mathcal{T}(\sigma,\sigma)$ onto the unit in $\mathcal{S}(\Phi(\sigma),\Phi(\sigma))$. In general, as the operators $\Phi : \mathcal{T}(\sigma,\sigma) \to \mathcal{S}(\Phi(\sigma),\Phi(\sigma))$ are *-homomorphisms of C^*-algebras, using C^*-equality, one gets

Proposition 2.9. *For any homomorphism $\Phi : \mathcal{T} \to \mathcal{S}$ of C^*-precategories the operators*

$$\Phi : \mathcal{T}(\sigma,\rho) \to \mathcal{S}(\Phi(\sigma),\Phi(\rho)), \qquad \sigma,\rho \in \mathcal{T}$$

are contractions, and if they are injective then they are isometries.

Clearly, if $\Phi : \mathcal{T} \to \mathcal{S}$ is a homomorphism and $\mathcal{K}$ is an ideal in $\mathcal{S}$ the collection of sets

$$\Phi^{-1}(\mathcal{K})(\sigma,\rho) := \{a \in \mathcal{T}(\sigma,\rho) : \Phi(a) \in \mathcal{K}\},$$

which shall be referred to as the *preimage of* $\mathcal{K}$, is an ideal in $\mathcal{T}$. In particular, the ideal being preimage of the zero ideal will be denoted by $\ker\Phi$ and called a *kernel* of Φ.

Proposition 2.10. *If $\mathcal{K}$ is an ideal in a C^*-precategory $\mathcal{T}$, then the precategory $\mathcal{T}/\mathcal{K}$ whose objects are the same as objects in $\mathcal{T}$ and morphisms are given by the quotient spaces*

$$(\mathcal{T}/\mathcal{K})(\sigma,\rho) := \mathcal{T}(\sigma,\rho)/\mathcal{K}(\sigma,\rho)$$

is a C^*-precategory and the quotient maps $q_{\mathcal{J}} : \mathcal{T}(\sigma,\rho) \to (\mathcal{T}/\mathcal{K})(\sigma,\rho)$ give rise to the quotient homomorphism of C^*-precategories $q_{\mathcal{J}} : \mathcal{T} \to \mathcal{T}/\mathcal{K}$.

Proof. Mimic the argument that shows the corresponding fact for C^*-algebras. ■

Definition 2.11. By a *representation of a C^*-precategory $\mathcal{T}$* in a C^*-algebra B we mean a homomorphism $\Phi : \mathcal{T} \to B$ where B is considered as a C^*-precategory with a single object. Equivalently π may be treated as a collection $\{\pi_{\sigma\rho}\}_{\sigma,\rho\in\mathcal{T}}$ of linear operators $\pi_{\sigma\rho} : \mathcal{T}(\rho,\sigma) \to B$ such that

$$\pi_{\sigma\rho}(a)^* = \pi_{\rho\sigma}(a^*), \quad \text{and} \quad \pi_{\tau\rho}(ba) = \pi_{\tau\sigma}(b)\pi_{\sigma\rho}(a),$$

for $a \in \mathcal{T}(\rho,\sigma)$, $b \in \mathcal{T}(\sigma,\tau)$. By a *representation* of $\mathcal{T}$ *in a Hilbert space* H we shall mean a representation of $\mathcal{T}$ in the C^*-algebra $L(H)$ of all bounded operators. We shall say that the representation is faithful if all the mappings $\{\pi_{\sigma\rho}\}_{\sigma,\rho\in\mathcal{T}}$ are injective.

Remark 2.12. The above definition differs from the one presented in [14, Def. 1.8]. However, as in [14, Prop. 1.14] (which can be easily refined to deal with C^*-precategories) one can show that for every C^*-precategory $\mathcal{T}$ there is a faithful representation of $\mathcal{T}$ in a Hilbert space.

Example 2.13 (C^*-category of Hilbert modules)**.** Let $\{X_\rho\}_{\rho\in\mathcal{T}}$ be the family of right Hilbert modules over a C^*-algebra A, indexed by a collection of objects $\mathcal{T}$. Then $\mathcal{T}$ with morphisms being the adjointable maps between the Hilbert A-modules:

$$\mathcal{T}(\rho,\sigma) := \mathcal{L}(X_\rho, X_\sigma), \qquad \rho,\, \sigma \in \mathcal{T},$$

becomes a C^*-category. The collection of spaces of "compact" operators

$$\mathcal{K}(\sigma,\rho) := \mathcal{K}(X_\rho, X_\sigma), \qquad \rho, \sigma \in \mathcal{T}.$$

give rise to the *ideal* $\mathcal{K}$ *of "compact operators"* in $\mathcal{T}$. Moreover, using Corollary 1.3, we see that every ideal J in A give naturally rise to two ideals in $\mathcal{T}$. One, denoted by $\mathcal{J}$, consists of all the adjointable maps with ranges in the spaces $X_\rho J$, $\rho\in\mathcal{T}$:

$$\mathcal{J}(\sigma,\rho) := \mathcal{L}(X_\rho, X_\sigma J) \cap \mathcal{L}(X_\rho, X_\sigma), \qquad \rho, \sigma \in \mathcal{T}.$$

The other one, being the intersection of $\mathcal{J}$ and $\mathcal{K}$, consists of all the "compact" maps with ranges in the spaces $X_\rho J$, $\rho\in\mathcal{T}$:

$$(\mathcal{K}\cap\mathcal{J})(\sigma,\rho) := \mathcal{K}(X_\rho, X_\sigma J), \qquad \rho, \sigma \in \mathcal{T}.$$

2.1. **C^*-precategory $\mathcal{T}_X$ of a C^*-correspondence.** Here we examine our model example of a C^*-precategory, which shall be equipped with an additional structure of a right tensor C^*-precategory in Subsection 3.2. Throughout this section we fix a C^*-correspondence X over a C^*-algebra A.

Definition 2.14. We define a *C^*-precategory $\mathcal{T}_X$ of the C^*-correspondence* X to be the C^*-precategory whose objects are natural numbers $\mathbb{N} = \{0, 1, ...\}$ and spaces of morphisms are

$$\mathcal{T}_X(0,0) := A, \qquad \mathcal{T}_X(0,m) := X^{\otimes m}, \qquad \mathcal{T}(X)(n,0) := \widetilde{X}^{\otimes n}$$
$$\mathcal{T}_X(n,m) := \mathcal{L}(X^{\otimes n}, X^{\otimes m}), \qquad \text{for } \ n,m\geq 1.$$

where $X^{\otimes n} := X\otimes\cdots\otimes X$ denotes the n-fold internal tensor product, $X^0 := A$, and $\widetilde{X}$ is the *dual module* to X. By Riesz-Fréchet theorem [31, Lem. 2.32] we may assume the following identifications

$$\mathcal{K}(A, X^{\otimes n}) = X^{\otimes n}, \qquad \mathcal{K}(X^{\otimes n}, A) = \widetilde{X}^{\otimes n},$$

which allow us to define the composition of morphisms in $\mathcal{T}_X$ as a composition of adjointable maps between Hilbert modules.

Obviously $\mathcal{T}_X$ may be treated as a sub-C^*-precategory in the C^*-category $\mathcal{T} = \{\mathcal{L}(X^{\otimes n}, X^{\otimes m})\}_{n,m\in\mathbb{N}}$, and if A is unital, then actually $\mathcal{T}_X = \mathcal{T}$. The reason why we deal with $\mathcal{T}_X$ rather than $\mathcal{T}$ is explained in Remark 3.3. Likewise in Example 2.13 we associate with an ideal J in A two ideals in $\mathcal{T}_X$.

Definition 2.15. We denote by $\mathcal{K}_X := \{\mathcal{K}(X^{\otimes n}, X^{\otimes m})\}_{n,m\in\mathbb{N}}$ the ideal in $\mathcal{T}_X$ consisting of "compact" operators, and for an ideal J l in A we put

$$\mathcal{K}_X(J) := \{\mathcal{K}(X^{\otimes n}, X^{\otimes m}J)\}_{n,m\in\mathbb{N}}, \qquad \mathcal{T}_X(J) := \{\mathcal{L}(X^{\otimes n}, X^{\otimes m}J)\cap\mathcal{T}_X(n,m)\}_{n,m\in\mathbb{N}}.$$

In particular, $\mathcal{K}_X(J)$ and $\mathcal{T}_X(J)$ are ideals in $\mathcal{T}_X$, $\mathcal{T}_X = \mathcal{T}_X(A)$ and $\mathcal{K}_X = \mathcal{K}_X(A)$.

These ideals give useful estimates for arbitrary ideals in $\mathcal{T}_X$.

Proposition 2.16. *Let $\mathcal{J}$ be an ideal in $\mathcal{T}_X$ and put $J := \mathcal{J}(0,0)$. Then*

$$\mathcal{K}_X(J) \subset \mathcal{J} \subset \mathcal{T}_X(J).$$

In particular, the relations

$$J = \mathcal{J}(0,0), \qquad \mathcal{J} = \mathcal{K}_X(J) \tag{8}$$

establish a one-to-one correspondence between ideals J in A and ideals $\mathcal{J}$ in $\mathcal{K}_X$.

Proof. By Theorem 2.6 an element $x \in X^{\otimes n} = \mathcal{T}_X(0,n)$ belongs to $\mathcal{J}(0,n)$ iff $x^*x = \langle x, x\rangle_A \in J$. Thus, by Hewitt-Cohen Factorization Theorem, we have $\mathcal{J}(0,n) = X^{\otimes n}J$. Since for $x \in \mathcal{J}(0,m) = X^{\otimes m}J$ and $y \in \mathcal{T}_X(0,n) = X^{\otimes n}$ we identify $xy^* \in \mathcal{J}(n,m)$ with the "one-dimensional" operator $\Theta_{x,y} \in \mathcal{K}(X^{\otimes n}, X^{\otimes m}J)$, one sees that $\mathcal{K}(X^{\otimes n}, X^{\otimes m}J) \subset \mathcal{J}(n,m)$ and consequently $\mathcal{K}_X(J) \subset \mathcal{J}$. To prove that $\mathcal{J} \subset \mathcal{T}_X(J)$ let $a \in \mathcal{J}(n,m) \subset \mathcal{L}(X^{\otimes n}, X^{\otimes m})$. Since for arbitrary $x \in \mathcal{T}(0,n) = X^{\otimes n}$ and $y \in \mathcal{T}(0,m) = X^{\otimes m}$ we have

$$x^*a^*y = \langle ax, y\rangle_A \in J,$$

it follows that a takes values in $X^{\otimes m}J$. ■

Corollary 2.17. *For any ideal I in A relations* (8) *establish a one-to-one correspondence between ideals J in I and ideals $\mathcal{J}$ in $\mathcal{K}_X(I)$.*

Proof. Use the transitivity of relation of being an ideal. ■

Remark 2.18. For an ideal I in A, XI is naturally considered as a C^*-correspondence over I. Applying Proposition 2.16 to XI we get a one-to-one correspondence between ideals J in I and ideals $\mathcal{J}$ in $\mathcal{K}_{XI}$, given by relations

$$J = \mathcal{J}(0,0), \qquad \mathcal{J} = \mathcal{K}_{XI}(J).$$

Thus by Corollary 2.17 the ideal structures of $\mathcal{K}_{XI}$ and $\mathcal{K}_X(I)$ are isomorphic. We note however that $\mathcal{K}_{XI} = \{\mathcal{K}(XI^{\otimes n}, XI^{\otimes m})\}_{n,m\in\mathbb{N}} \neq \mathcal{K}_X(I) = \{\mathcal{K}(X^{\otimes n}, X^{\otimes m}I)\}_{n,m\in\mathbb{N}}$ unless $\phi(I)X = XI$. The consequences of this phenomena will be pursued in Subsection 6.1.

We illustrate the introduced objects in the context of C^*-correspondences from Examples 1.11, 1.12.

Example 2.19 (C^*-precategory of a partial morphism)**.** Let X_φ be the C^*-correspondence of a partial morphism $\varphi : A \to M(A_0)$, cf. Example 1.11. We shall call $\mathcal{T}_\varphi := \mathcal{T}_{X_\varphi}$ a *C^*-precategory of φ*. We have the following natural identifications for the ideal $\mathcal{K}_\varphi := \mathcal{K}_{X_\varphi} = \{\mathcal{K}(X_\varphi^{\otimes n}, X_\varphi^{\otimes m})\}_{n,m\in\mathbb{N}}$:

$$\mathcal{K}_\varphi(n,m) = \varphi\Big(\varphi\big(...\underbrace{\varphi(A_0)A_0)...\big)\Big)A_0}_{m}\, A\, \varphi\Big(\varphi\big(...\underbrace{\varphi(A_0)A_0)...\big)\Big)A_0}_{n}, \tag{9}$$

$n, m \in \mathbb{N}$, and in view of Proposition 2.16 every ideal $\mathcal{J}$ in $\mathcal{K}_\varphi$ is of the form

$$\mathcal{J}(n,m) = \varphi\Big(\varphi\big(...\underbrace{\varphi(A_0)A_0)...\big)\Big)A_0}_{m}\, J\, \varphi\Big(\varphi\big(...\underbrace{\varphi(A_0)A_0)...\big)\Big)A_0}_{n}, \qquad n, m \in \mathbb{N},$$

where $J = \mathcal{J}(0,0)$ is an ideal in A. In particular, if φ is a partial morphism arising from a partial automorphism (θ, I, A_0), then denoting by D_n the domain of θ^{-n}, cf. [9], the above ideals are given by

$$\mathcal{K}_\varphi(n,m) = D_{\max\{m,n\}}, \qquad \mathcal{J}(n,m) = J \cap D_{\max\{m,n\}}, \qquad n, m \in \mathbb{N}.$$

If in turn φ arise from an endomorphism $\alpha : A \to A$, we then have

$$\mathcal{K}_\varphi(n,m) = \alpha^m(A)A\alpha^n(A), \qquad \mathcal{J}(n,m) = \alpha^m(A)J\alpha^n(A), \qquad n, m \in \mathbb{N}.$$

In this event we shall denote the C^*-precategory $\mathcal{T}_\varphi$ by $\mathcal{T}_\alpha$.

Example 2.20 (C^*-precategory of a directed graph)**.** We set $\mathcal{T}_E := \mathcal{T}_{X_E}$ where X_E is a C^*-correspondence associated with a directed graph $E = (E^0, E^1, r, s)$, cf. Example 1.12, and call it a *C^*-precategory of the directed graph E*. For $n \geq 1$ we denote by E^n the set of all paths of length n, i.e. the set of sequences $(e_1, e_2, ..., e_n)$ where $e_i \in E^1$ and $r(e_i) = s(e_{i+1})$, and we put $r^{(n)}(e_1, e_2, ..., e_n) := r(e_n)$, $s^{(n)}(e_1, e_2, ..., e_n) := s(e_n)$. Then $E^{(n)} := (E^0, E^n, r^{(n)}, s^{(n)})$ is a directed graph and the C^*-correspondence $X_{E^{(n)}}$ may be naturally identified with $X_E^{\otimes n}$. In particular, $X_E^{\otimes n}$ is spanned by the point masses $\{\delta_\mu : \mu \in E^n\}$ and the ideal of "compact" operators $\mathcal{K}_E := \mathcal{K}_{X_E}$ is spanned by the "matrix units"

$$(10) \qquad \Theta_{\delta_\mu, \delta_\nu} \qquad \text{such that} \qquad r^{(m)}(\mu) = r^{(n)}(\nu), \qquad \mu \in E^m, \nu \in E^n.$$

Since every ideal in $A = C_0(E^0)$ is determined by its hull in E^0, in view of Proposition 2.16, the equalities, for $n, m \in \mathbb{N}$,

$$(11) \qquad \mathcal{J}(n,m) = \overline{\text{span}}\{\Theta_{\delta_\mu, \delta_\nu} : r^{(m)}(\mu) = r^{(n)}(\nu) \notin V, \mu \in E^m, \nu \in E^n\}.$$

establish a one-to-one correspondence between subsets V of E^0 and ideals $\mathcal{J}$ in $\mathcal{K}_E$.

3. Right tensor C^*-precategories and their representations

As it was noticed in the introduction the following definition generalizes the notion of a (strict) tensor C^*-category, cf. [12], in the case the underlying semigroup is $\mathbb{N}$.

Definition 3.1. By a *right tensor C^*-precategory* we will mean a C^*-precategory $\mathcal{T}$ with the set of objects $\mathbb{N} = \{0, 1, 2, ...\}$, and with a designated endomorphism $\otimes 1 : \mathcal{T} \to \mathcal{T}$ sending n to $n+1$:

$$\otimes 1 : \mathcal{T}(n,m) \to \mathcal{T}(n+1, m+1).$$

We will call $\otimes 1$ a *right tensoring* on $\mathcal{T}$ and write $a \otimes 1$ instead of $\otimes 1(a)$ for a morphism a in $\mathcal{T}$. If $\mathcal{T}$ is a C^*-category we shall refer to it as a *right tensor C^*-category.*

Iterating a right tensoring $\otimes 1$ on a C^*-precategory $\mathcal{T}$ one gets the semigroup $\{\otimes 1^k\}_{k \in \mathbb{N}}$ of endomorphisms $\otimes 1^k : \mathcal{T} \to \mathcal{T}$

$$\otimes 1^k : \mathcal{T}(n,m) \to \mathcal{T}(n+k, m+k),$$

where by convention we put $\otimes 1^0 := id$. The model example of a right-tensor C^*-precategory is the C^*-precategory $\mathcal{T}_X$ with a right tensoring induced by the homomorphism $\phi : A \to \mathcal{L}(X)$ (cf. [13], [17, Defn. 1.6, 1.7]), which we now describe in detail.

3.2. Right tensor C^*-precategory $\mathcal{T}_X$ of a C^*-correspondence X. Let $\mathcal{T}_X$ be as in Definition 2.14. If $n > 0, m > 0$, we have a natural tensoring

$$\mathcal{T}_X(n,m) = \mathcal{L}(X^{\otimes n}, X^{\otimes m}) \ni a \mapsto a \otimes 1 \in \mathcal{L}(X^{\otimes(n+1)}, X^{\otimes(m+1)}) = \mathcal{T}_X(n+1, m+1),$$

given by (2). For $a \in \mathcal{T}_X(0,0) = A$ we put $a \otimes 1 := \phi(a)$. In order to define the right tensoring on the spaces $\mathcal{T}_X(0,n)$, $\mathcal{T}_X(n,0)$, for $n > 0$, we use the mappings $L^{(n)} : X^{\otimes n} \mapsto \mathcal{L}(X, X^{\otimes n+1})$ and $D^{(n)} : \widetilde{X}^{\otimes n} \mapsto \mathcal{L}(X^{\otimes n+1}, X)$, determined by the formulas

$$[L^{(n)}(x)](y) := x \otimes y, \qquad D^{(n)}(\flat(x)) := (L^{(n)}(x))^* \qquad x \in X^{\otimes n}, y \in X,$$

where $\flat : X \to \widetilde{X}$ is the canonical anti-linear isomorphism. In particular, for $y_1 \in X^{\otimes(n)}$, $y_2 \in X$, we have $[D^{(n)}(\flat(x))](y_1 \otimes y_2) = \phi(\langle x, y_1 \rangle_A) y_2$. We adopt the following notation

$$a \otimes 1 := L^{(n)}(a), \qquad a \in \mathcal{T}_X(0,n) = X^{\otimes n},$$

$$a \otimes 1 := D^{(n)}(a), \qquad a \in \mathcal{T}_X(n,0) = \widetilde{X}^{\otimes n}.$$

In this way $\mathcal{T}_X$ becomes a right tensor C^*-precategory with right tensoring "$\otimes 1$". We shall call it a *right tensor C^*-precategory of the C^*-correspondence X.*

Remark 3.3. The *-homomorphism $\varphi : A \to \mathcal{L}(X)$ need not extend to a *-homomorphism $M(A) \to \mathcal{L}(X)$ unless it is nondegenerate (or A is unital). In other words, in general there is no obvious right tensoring on the C^*-category $\mathcal{T} := \{\mathcal{L}(X^{\otimes n}, X^{\otimes m})\}_{n,m\in\mathbb{N}}$ and thus the sub-C^*-precategory $\mathcal{T}_X \subset \mathcal{T}$ seems to be a more appropriate object to work with.

Applying the above construction to Examples 2.19, 2.20 we get respectively a *right tensor C^*-precategory $\mathcal{T}_\varphi$ of a partial morphism φ* and a *right tensor C^*-precategory $\mathcal{T}_E$ of a directed graph E.*

Example 3.4 (Right tensor C^*-precategory of a partial morphism)**.** The C^*-precategory $\mathcal{T}_\varphi$ from Example 2.19 is a right tensor C^*-precategory with right tensoring induced by φ. In particular, if φ arises from an endomorphism $\alpha : A \to A$ of a C^*-algebra A, the ideal $\mathcal{K}_\varphi = \{\alpha^m(A)A\alpha^n(A)\}_{n,m\in\mathbb{N}}$ is a right tensor C^*-precategory itself and the right tensoring on $\mathcal{K}_\varphi$ assumes the form

$$\alpha^m(A)A\alpha^n(A) \ni a \longrightarrow a \otimes 1 = \alpha(a) \in \alpha^{m+1}(A)A\alpha^{n+1}(A).$$

If additionally A is unital the above formula describes a right tensoring on $\mathcal{T}_\alpha = \mathcal{T}_\varphi$ as then we have $\mathcal{T}_\alpha = \mathcal{K}_\varphi = \{\alpha^m(1)A\alpha^n(1)\}_{n,m\in\mathbb{N}}$.

Example 3.5 (Right tensor C^*-precategory of a directed graph)**.** If $E = (E^0, E^1, r, s)$ is a directed graph, then $\mathcal{T}_E$, see Example 2.20, is a right tensor C^*-precategory where the right tensoring is induced by "composition" of graphs or equivalently by multiplication of the incidence matrices. In the event the set of egdes E^1 is finite, the C^*-precategory $\mathcal{T}_E$ coincides with the ideal of "compact" operators $\mathcal{K}_E$ and the right tensoring is determined by the formula

$$\Theta_{\delta_\mu,\delta_\nu} \otimes 1 := \sum_{\{e\in E^1:\, r^{(m)}(\mu)=s(e)\}} \Theta_{\delta_{\mu e},\delta_{\nu e}}\,, \qquad \mu \in E^m, \nu \in E^n,$$

where μe and νe are the paths obtained by concatenation, cf. Example 2.20.

Now, we now turn to an investigation of representations of ideals in right tensor C^*-precategories which respect the right tensoring. As we shall see these representations generalize representations of C^*-correspondences.

Definition 3.6. Let $\mathcal{K}$ be an ideal in a right tensor C^*-precategory $\mathcal{T}$. We will say that a representation $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$, cf. Definition 2.11, is a *right tensor representation* if it satisfies

$$\pi_{nm}(a)\pi_{m+k,l}(b) = \pi_{n+k,l}((a \otimes 1^k)\, b) \tag{12}$$

for all $a \in \mathcal{K}(m,n)$ and $b \in \mathcal{K}(l, m+k)$, $k,l,m,n \in \mathbb{N}$.

Remark 3.7. Since $\mathcal{K}$ is an ideal the right hand side of (12) makes sense. Furthermore, by taking adjoints one gets the symmetrized version of this equation:

$$\pi_{n,m+k}(a)\pi_{ml}(b) = \pi_{n,l+k}(a(b \otimes 1^k)),$$

where $a \in \mathcal{K}(m+k,n)$ and $b \in \mathcal{K}(l,m)$, $k,l,m,n \in \mathbb{N}$.

The following statement generalizes an elementary, but frequently used, fact that a representation of an ideal in C^*-algebra extends to a representation of this algebra.

Proposition 3.8. *Suppose that $\mathcal{K}$ is an ideal in $\mathcal{T}$ and $\pi = \{\pi_{nm}\}_{n,m\in\mathbb{N}}$ is a right tensor representation of $\mathcal{K}$ in a Hilbert space H. If we denote by P_m, $m \in \mathbb{N}$, the orthogonal projection onto the essential subspaces for π_{mm}:*

$$P_m H = \pi_{mm}(\mathcal{T}(m,m))H,$$

then

i) *projections P_m, $m \in \mathbb{N}$, are pairwisely commuting and essential subspaces of π_{nm}, $n \in \mathbb{N}$, are contained in $P_m H$.*

ii) *There is a unique extension $\overline{\pi} = \{\overline{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ of π to a representation of a C^*-precategory $\mathcal{T}$ such that essential subspaces of $\overline{\pi}_{nm}$, $n \in \mathbb{N}$, are contained in $P_m H$. This extension is a right tensor representation of $\mathcal{T}$ and it is determined by relations*

$$\overline{\pi}_{n,m}(a)\pi_{m,m}(b)h = \pi_{n,m}(ab)h, \qquad a \in \mathcal{T}(m,n),\, b \in \mathcal{K}(m,m), h \in H. \tag{13}$$

iii) *For $a \in \mathcal{T}(m,n)$, $k \in \mathbb{N}$, and objects from i), ii) we have*

$$\overline{\pi}_{n,m}(a)P_{m+k} = \overline{\pi}_{n+k,m+k}(a \otimes 1^k), \qquad P_{n+k}\overline{\pi}_{n,m}(a) = \overline{\pi}_{n+k,m+k}(a \otimes 1^k).$$

Proof. i). Fix $m \in \mathbb{N}$ and let $\{\mu_\lambda\}$ be an approximate unit for $\mathcal{K}(m,m)$. If $x \in H$, $x \neq 0$, is orthogonal to $P_m H$, then using Lemma 2.5, for every $a \in \mathcal{K}(m,n)$, $n \in \mathbb{N}$ and $y \in H$ we get

$$\langle \pi_{nm}(a)x, y\rangle = \langle x, \pi_{mn}(a^*)y\rangle = \lim_\lambda \langle x, \pi_{mm}(\mu_\lambda)\, \pi_{mn}(a^*)y\rangle = 0.$$

Thus $\pi_{nm}(a)x = 0$ and x is not in the essential subspace of representation π_{nm}. To see that projections P_m, P_{m+k}, $k \in \mathbb{N}$, commute let now $\{\mu_\lambda\}$ be an approximate unit for $\mathcal{K}(m+k, m+k)$. Then for $a \in \mathcal{K}(m,m)$ and $h \in H$ we have

$$\begin{aligned} P_{m+k}\pi_{mm}(a)h &= \lim_\lambda \pi_{m+k,m+k}(\mu_\lambda)\pi_{mm}(a)h = \lim_\lambda \pi_{m+k,m+k}(\mu_\lambda(a \otimes 1^k))h \\ &= \lim_\lambda \pi_{m+k,m+k}((a \otimes 1^k)\mu_\lambda)h = \lim_\lambda \pi_{mm}(a)\pi_{m+k,m+k}(\mu_\lambda)h \\ &= \pi_{mm}(a)P_{m+k}h, \end{aligned}$$

that is $P_{m+k}\pi_{mm}(a) = \pi_{mm}(a)P_{m+k}$ and consequently $P_{m+k}P_m = P_{m+k}P_m$.
ii). Formula (13) give rise to the representation $\overline{\pi}_{nm}$ of $\mathcal{T}(m,n)$ on $P_m H$ since for $a \in \mathcal{T}(m,n)$, $b \in \mathcal{K}(m,m), h \in H$, and an approximate unit $\{\mu_\lambda\}$ for $\mathcal{K}(m,m)$ we have

$$\begin{aligned} \|\pi_{n,m}(ab)h\| &= \lim_\lambda \|\pi_{n,m}(a\mu_\lambda)\pi_{m,m}(b)h\| \leq \lim_\lambda \|\pi_{n,m}(a\mu_\lambda)\|\|\pi_{m,m}(b)h\| \\ &\leq \|a\|\|\pi_{m,m}(b)h\|. \end{aligned}$$

Defining $\overline{\pi}_{nm}$ to be zero on $(P_m H)^\perp$ one readily sees that $\overline{\pi} = \{\overline{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ is a representation of the C^*-precategory $\mathcal{T}$. Obviously, every representation $\overline{\pi} = \{\overline{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ that extends π satisfies (13). This together with the requirement that $\overline{\pi}_{nm}|_{(P_m H)^\perp} \equiv 0$, for all $m, n \in \mathbb{N}$, determines $\overline{\pi}$ uniquely.
iii). Since π is a right tensor representation, actions of $\overline{\pi}_{n,m}(a \otimes 1^k)$ and $\overline{\pi}_{n+k,m+k}(a)$ on an element of the form $\pi_{m,m}(b)h$ coincide. Hence $\overline{\pi}_{n,m}(a)P_{m+k} = \overline{\pi}_{n+k,m+k}(a\otimes 1^k)$. By passing to adjoints one gets $P_{n+k}\overline{\pi}_{n,m}(a) = \overline{\pi}_{n+k,m+k}(a \otimes 1^k)$. ■

Corollary 3.9. *We have a one-to-one correspondence between right tensor representations of the ideal $\mathcal{K}$ in the Hilbert space H and right tensor representations $\{\overline{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ of $\mathcal{T}$ in H satisfying*

$$\overline{\pi}_{mm}(\mathcal{T}(m,m))H = \overline{\pi}_{mm}(\mathcal{K}(m,m))H, \qquad m \in \mathbb{N}. \tag{14}$$

Proof. Clear by Proposition 3.8 and Lemma 2.5. ■

3.1. **Right tensor representations induced by representations of C^*-correspondences.** Let X be a C^*-correspondence over A. We shall investigate relationships between right tensor representations of the right tensor C^*-precategory $\mathcal{T}_X$, its ideal $\mathcal{K}_X = \{\mathcal{K}(X^{\otimes n}, X^{\otimes m})\}_{n,m\in\mathbb{N}}$, and representations of X.

Definition 3.10. A *representation* (π, t) of the C^*-correspondence X in a C^*-algebra B consists of a linear map $t : X \to B$ and a *-homomorphism $\pi : A \to B$ such that

$$t(x \cdot a) = t(x)\pi(a), \quad t(x)^*t(y) = \pi(\langle x, y\rangle_A), \quad t(a \cdot x) = \pi(a)t(x),$$

for $x, y \in X$ and $a \in A$. If π is faithful (then automatically t is isometric, cf. [11], [29]) we say that the representation (π, t) is *faithful*. If $B = L(H)$ for a Hilbert space H we say that (π, t) is a *representation of X on the Hilbert space H*.

Remark 3.11. The above introduced notion is called a *Toeplitz representation* of X in [11], [10], and an *isometric covariant representation* of X in [28].

The first step is to show that representation (π, t) give rise to a right tensor representation of $\mathcal{K}_X$ and this, in essence, follows from the results of [15], [10], [11], [30] where it was used in an implicit form.

Theorem 3.12. *If (π, t) is a representation of X in a C^*-algebra B, then there is a unique right tensor representation $\{\pi_{nm}\}_{m,n\in\mathbb{N}}$ of the ideal $\mathcal{K}_X$ in the right tensor C^*-precategory $\mathcal{T}_X$, such that*

$$\pi_{00} = \pi, \qquad \pi_{10} = t. \tag{15}$$

We shall denote this representation by $[\pi, t]$. In particular, every right tensor representation of $\mathcal{K}_X$ is of the form $[\pi, t]$ and

$$\ker[\pi, t] = \mathcal{K}_X(\ker \pi),$$

see Definition 2.15.

Proof. Suppose that $\{\pi_{nm}\}_{m,n\in\mathbb{N}}$ is a right tensor representation of $\mathcal{K}_X$ such that (15) holds. Then in view of Definition 3.6, for $x_i \in X$, $i = 1, ..., n$, we have

$$\pi_{n0}(x_1 \otimes \cdots \otimes x_n) = t(x_1)t(x_2)\cdots t(x_n). \tag{16}$$

Hence π_{n0} is uniquely determined by t. Furthermore π_{nm} is uniquely determined by π_{n0} and π_{m0}, since for $x \in X^{\otimes n}$ and $y \in X^{\otimes m}$ we have

$$\pi_{nm}(\Theta_{x,y}) = \pi_{n0}(x)\pi_{0m}(y)^*. \tag{17}$$

This proves the uniqueness of the representation $[\pi, t] = \{\pi_{nm}\}_{n,m\in\mathbb{N}}$. For the existence of $[\pi, t]$ note that formula (16) give rise to the linear map $\pi_{n0} : X^{\otimes n} \to B$ such that the pair (π_{n0}, π) is a representation of $X^{\otimes n}$, cf. [11, Proposition 1.8] or [10, Lemma 3.6]. Arguing as in [15, Lemma 2.2] or [30, Lemma 3.2], one sees that formula (17) defines a contraction $\pi_{nm} : \mathcal{K}(X^{\otimes m}, X^{\otimes n}) \to B$ such that the family $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ forms a representation of $\mathcal{K}_X$ in B. To check that $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ is a right tensor representation let $x = \otimes_{i=1}^n x_i \in X^{\otimes n}$, $y = \otimes_{i=1}^m y_i \in X^{\otimes m}$, $x' = \otimes_{i=1}^{m+k} x'_i \in X^{\otimes m+k}$ and $y' = \otimes_{i=1}^{l} y'_i \in X^{\otimes l}$ (we adhere to the convention that the indexes indicate the order of factors). One readily sees that

$$(\Theta_{x,y} \otimes 1^k)\Theta_{x',y'} = \Theta_{z,y'}$$

where $z = x\langle \otimes_{i=1}^{m} y_i, \otimes_{i=1}^{m} x_i' \rangle \otimes x_{m+1}' \otimes ... \otimes x_{m+k}' \in X^{\otimes m+k}$. On the other hand

$$\begin{aligned}\pi_{nm}(\Theta_{x,y})\pi_{m+k,l}(\Theta_{x',y'}) &= \prod_{i=1}^{n} t(x_i)\Big(\prod_{i=1}^{m} t(y_i)\Big)^* \prod_{i=1}^{m+k} t(x_i')\Big(\prod_{i=1}^{l} t(y_i')\Big)^* \\ &= \prod_{i=1}^{n} t(x_i)\pi(\Big\langle \otimes_{i=1}^{m} y_i, \otimes_{i=1}^{m} x_i' \Big\rangle_A) \prod_{i=1}^{k} t(x_{m+i}')\Big(\prod_{i=1}^{l} t(y_i')\Big)^* \\ &= \pi_{n+k,m+k}(\Theta_{z,y'}),\end{aligned}$$

which means that condition (12) is satisfied by "rank one" operators and thereby by all morphisms of the ideal $\mathcal{K}_X$.
Clearly, if $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ is a right tensor representation of $\mathcal{K}_X$, then the pair (π, t) where $\pi_{00} := \pi$, $t := \pi_{10}$, is a representation of X and we have $[\pi, t] = \{\pi_{nm}\}_{n,m\in\mathbb{N}}$. To investigate the form of $\ker \pi_{nm}$ note that for $x \in X^{\otimes n}$ we have

$$\|\pi_{n0}(x)\|^2 = \|\pi_{n0}(x)^*\pi_{n0}(x)\| = \|\pi(\langle x, x\rangle_A)\| \le \|\langle x, x\rangle_A\| = \|x\|^2.$$

Hence, it follows that $\pi_{n0}(x) = 0 \iff \langle x, x\rangle_A \in \ker \pi$. Thereby, by Hewitt-Cohen Factorization Theorem, $\ker \pi_{n0} = X^{\otimes n} \ker \pi$. In view of formulas (16), (17) $a \in \mathcal{K}(X^{\otimes m}, X^{\otimes n})$ belongs to $\ker \pi_{nm}$ iff the range of a is contained in $\ker \pi_{n0} = X^{\otimes n} \ker \pi$. ∎

Remark 3.13. The mapping π_{11} in the above theorem is determined by the formula

$$\pi_{11}(\Theta_{x,y}) = t(x)t(y)^*, \qquad x, y \in X.$$

In [28], [10],[11], [29], it is denoted by $\pi^{(1)}$ where it is used to introduce the notion of coisometricity, cf. Definition 3.20 below.

If (π, t) is a representation of X on a Hilbert space H, then within the notation of Proposition 3.12, the essential subspace for π_{mm}, $m = 1, 2, ...,$ is

$$\pi_{mm}(\mathcal{K}(X^{\otimes m}))H = \overline{\text{span}}\{t(x_1) \cdot ... \cdot t(x_m)h : x_1, ..., x_m \in X,\ h \in H\}. \tag{18}$$

Hence the projections P_m onto these spaces, associated to $[\pi, t]$ via Propositions 3.8, form a decreasing sequence: $P_1 \ge P_2 \ge ...$, cf. [11, Prop. 1.6], [10, 4.3]. As an immediate consequence of Propositions 3.12, 3.8 we get

Theorem 3.14. *Every representation (π, t) of a C^*-correspondence X in a Hilbert space H give rise to a unique right tensor representation $\{\overline{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ of $\mathcal{T}_X$ such that*

$$\overline{\pi}_{00} = \pi, \qquad \overline{\pi}_{01} = t. \tag{19}$$

and

$$\overline{\pi}_{mm}(\mathcal{L}(X^{\otimes m}))H = \overline{\pi}_{mm}(\mathcal{K}(X^{\otimes m}))H, \qquad m = 1, 2, ... \tag{20}$$

We shall denote this representation by $\overline{[\pi, t]}$. An arbitrary right tensor representation of $\mathcal{T}_X$ in a Hilbert space is of the form $\overline{[\pi, t]}$ if and only if it satisfies (20)*. Moreover, the kernel of $\overline{[\pi, t]}$ is determined by the kernel of π:*

$$\ker \overline{[\pi, t]} = \mathcal{T}_X(\ker \pi),$$

see Definition 2.15.

Proof. The only thing requiring a comment is the form of $\ker \overline{[\pi, t]}$ but this follows from (13) and the equality $\ker[\pi, t] = \mathcal{K}_X(\ker \pi)$. ∎

Example 3.15. Let $\alpha : A \to A$ be an endomorphism of a unital C^*-algebra A. A triple (π, U, B) consisting of a *-homomorphism $\pi : A \to B$ and a partial isometry $U \in B$ such that

$$\pi(\alpha(a)) = U\pi(a)U^*, \quad \text{for all } a \in A, \quad \text{and} \quad U^*U \in \pi(A)',$$

is called a *covariant representation of* α *in* B, see [27], [24], [25]. It is known [10], [25] that covariant representations of α are in one-to-one correspondence with representations of the C^*-correspondence $X_\alpha = \alpha(1)A$ (defined in Example 1.11). Thus in view of Theorem 3.12 every right tensor representation of the right tensor C^*-category $\mathcal{T}_\alpha = \{\alpha^m(1)A\alpha^n(1)\}_{n,m\in\mathbb{N}}$, cf. Examples 2.19, 3.4, assumes the form

$$\pi_{nm}(a) := U^{*n}\pi(a)U^m, \qquad a \in \alpha^n(1)\mathcal{A}\alpha^m(1),\ n, m \in \mathbb{N},$$

for a covariant representation (π, U, B) of α where $\pi = \pi_{00}$ and $U = \pi_{01}(\alpha(1))$. In particular, the C^*-algebra generated by the image of the C^*-category $\mathcal{T}_\alpha$ under $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ is the C^*-algebra

$$C^*(\pi(A), U) = \overline{\text{span}}\{U^{*n}aU^m : a \in A, m, n \in \mathbb{N}\}$$

generated by $\pi(A)$ and U.

In a more general context we may define a *covariant representation of a partial morphism* $\varphi : A \to M(A_0)$ to be a triple (π, U, H) consisting of a Hilbert space H, a *-homomorphism $\pi : A \to L(H)$ and a partial isometry $U \in L(H)$ such that

$$U\pi(a)U^* = \overline{\pi}(\varphi(a)), \quad \text{for all } a \in A, \quad \text{and} \quad U^*U \in \pi(A)',$$

where $\overline{\pi} : M(A_0) \to H$ is a representation given by the conditions

$$\pi(b)|_{(\pi(A_0)H)^\perp} \equiv 0 \qquad \text{and} \qquad \overline{\pi}(b)a_0h = \pi(ba_0)h, \quad b \in M(A_0),\ a_0 \in A_0,\ h \in H.$$

It is straightforward to check that if (π, U, H) is a covariant representation of φ, then the pair (π, t) where

$$t(x) := U^*\pi(x), \qquad x \in X_\varphi = A_0A,$$

is a representation of the C^*-correspondence X_φ. Conversely, if (π, t) is a representation of X_φ in a Hilbert space H, then for $a_0 \in A_0$ and $h \in H$ we have

$$\|t(a_0)h\|^2 = \langle t(a_0)h, t(a_0)h\rangle = \langle \pi(a_0^*a_0)h, h\rangle = \langle \pi(a_0)h, \pi(a_0)h\rangle = \|\pi(a_0)h\|^2.$$

Thus relations

$$Ut(a_0)h = \pi(a_0)h, \qquad a_0 \in A_0,\ h \in H, \qquad \text{and} \quad U|_{(t(A_0)H)^\perp} \equiv 0$$

define a partial isometry $U \in L(H)$. For $a_0 \in A_0$, $a \in A$ and $h \in H$ we have

$$U\pi(a)U^*\pi(a_0)h = U\pi(a)t(a_0)h = Ut(\varphi(a)a_0)h = \pi(\varphi(a)a_0)h = \overline{\pi}(\varphi(a))\pi(a_0)h$$

and

$$\pi(a)U^*Ut(a_0)h = \pi(a)t(a_0)h = t(\varphi(a)a_0)h = U^*Ut(\varphi(a)a_0)h = U^*U\pi(a)t(a_0)h.$$

Hence the triple (π, U, H) is a covariant representation of φ. As a consequence, see Theorem 3.14, it follows that right tensor representations $\{\overline{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ of $\mathcal{T}_\varphi$ satisfying (20) are in one-to-one correspondence with covariant representations (π, U, H) of φ. In particular, the C^*-algebra generated by the image of the C^*-precategory $\mathcal{T}_\varphi$ under $\{\overline{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ is contained the W^*-algebra $W^*(\pi(A), U)$ generated by $\pi(A)$ and U.

Example 3.16. Let $\mathcal{T}_E$ be the right tensor C^*-precategory of a directed graph $E = (E^0, E^1, r, s)$. By a *Toeplitz-Cuntz-Krieger E-family* it is meant a collection of mutually orthogonal projections $\{p_v : v \in E^0\}$ together with a collection of partial isometries $\{s_e : e \in E^1\}$ that satisfy relations

$$s_e^* s_e = p_{r(e)}, \qquad s_e s_e^* < p_{s(e)}, \qquad e \in E^1.$$

Such families are in one-to-one correspondence with representations of X_E, cf. [16], [29], [7]. Hence by Theorem 3.12 every right tensor representation $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ of the ideal $\mathcal{K}_E = \{\mathcal{K}(X_E^{\otimes n}, X_E^{\otimes m})\}_{n,m\in\mathbb{N}}$ in $\mathcal{T}_E$, see Example 2.20, is given by the formula

$$\pi_{nm}(\Theta_{\delta_\mu,\delta_\nu}) = s_{e_1}...s_{e_m} s_{f_1}^*...s_{f_n}^* \qquad \mu = (e_1, ..., e_m) \in E^m, \nu = (e_1, ..., f_n) \in E^n,$$

for a Toeplitz-Cuntz-Krieger E-family $\{p_v : v \in E^0\}$, $\{s_e : e \in E^1\}$, where

$$p_v = \pi_{00}(\delta_v), \quad v \in E^0 \qquad \text{and} \qquad s_e = \pi_{01}(\delta_e), \quad e \in E^1.$$

In particular, the algebra generated by the image of the ideal $\mathcal{K}_E$ under representation $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ is the C^*-algebra

$$C^*(\{p_v : v \in E^0\}, \{s_e : e \in E^1\})$$

generated by the corresponding Toeplitz-Cuntz-Krieger E-family.

3.2. **Ideals of coisometricity for right tensor representations.** We shall now transfer the concept of coisometricity for representations of C^*-correspondences, introduced by Muhly and Solel in [28], onto the ground of ideals in right tensor C^*-precategories.

Definition 3.17. For an ideal $\mathcal{K}$ in a right tensor C^*-precategory $\mathcal{T}$ we put $J(\mathcal{K}) := (\otimes 1)^{-1}(\mathcal{K}) \cap \mathcal{K}$. In other words, $J(\mathcal{K}) = \{J(\mathcal{K})(n, m)\}_{n,m\in\mathbb{N}}$ is an ideal in $\mathcal{K}$ where

$$J(\mathcal{K})(n, m) := \big\{a \in \mathcal{K}(n, m) : a \otimes 1 \in \mathcal{K}(n + 1, m + 1)\big\}. \tag{21}$$

The ideal $J(\mathcal{K})$ plays the role of ideal $J(X) = \phi^{-1}(\mathcal{K}(X))$ introduced by Pimsner in [30]. Roughly speaking, it consists of the elements for which the notion of coisometricity makes sense.

Proposition 3.18. *Let $\mathcal{K}$ be an ideal in a right tensor C^*-precategory $\mathcal{T}$. If $\pi = \{\pi_{nm}\}_{n,m\in\mathbb{N}}$ is a right tensor representation of $\mathcal{K}$ in a C^*-algebra B, then the spaces*

$$\mathcal{J}(n, m) := \Big\{a \in J(\mathcal{K})(n, m) : \pi_{nm}(a) = \pi_{n+1,m+1}(a \otimes 1)\Big\}$$

form an ideal in $J(\mathcal{K})$. Moreover, we have $\mathcal{J} \cap (\ker \otimes 1) \subset \ker \pi$ and $\ker \pi \cap J(\mathcal{K}) \subset \mathcal{J}$. In particular, if the representation $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ is faithful, then $\mathcal{J} \subset (\ker \otimes 1)^\perp$.

Proof. The first part of proposition follows directly from the definition of right tensor representation. To show the second part, note that for $a \in \mathcal{J}(n, m) \cap \ker(\otimes 1)(n, m)$ we have $\pi_{nm}(a) = \pi_{n+1,m+1}(a \otimes 1) = \pi_{n+1,m+1}(0) = 0$, that is $a \in (\ker \pi)(n, m)$. If, in turn, $a \in \ker \pi(n, m) \cap J(\mathcal{K})(n, m)$, then considering π as a representation on Hilbert space and using Proposition 3.8 iii), we get $\pi_{n+1,m+1}(a \otimes 1) = \pi_{nm}(a)P_{m+1} = 0$ and hence $a \in \mathcal{J}(n, m)$. ∎

Definition 3.19. Let $\mathcal{K}$ be an ideal in a right tensor C^*-precategory $\mathcal{T}$ and let $\pi = \{\pi_{nm}\}_{n,m\in\mathbb{N}}$ be a right tensor representation of $\mathcal{K}$. We shall say that π is *coisometric* on an ideal $\mathcal{J} \subset J(\mathcal{K})$ if

$$\pi_{nm}(a) = \pi_{n+1,m+1}(a \otimes 1), \qquad \text{for all} \quad a \in \mathcal{J}(n, m),\ n, m \in \mathbb{N}.$$

The ideal $\mathcal{J}$ defined in Proposition 3.18 is the biggest ideal on which π is coisometric and we shall call it the *ideal of coisometricity for π*.

We devote the rest of this subsection to discuss and reveal the relationship between the above definition and the following one.

Definition 3.20 (cf. [10], [28])**.** A representation (π, t) of a C^*-correspondence X is called *coisometric* on an ideal J contained in $J(X) = \phi^{-1}(\mathcal{K}(X))$ if

$$\pi(a) = \pi_{11}(a \otimes 1), \qquad \text{for all } \ a \in J,$$

where π_{11} is defined in Remark 3.13. The set $\{a \in J(X) : \pi(a) = \pi_{11}(a \otimes 1)\}$ is the biggest ideal on which (π, t) is coisometric and we shall call it an *ideal of coisometricity for* (π, t).

As an immediate corollary of Propositions 2.16, 3.18 we get

Theorem 3.21. *If (π, t) is a representation of X in a C^*-algebra B and*

$$J = \{a \in J(X) : \pi_{11}(\phi(a)) = \pi(a)\} \tag{22}$$

then the ideal of coisometricity for the right tensor representation $[\pi, t]$ of the ideal $\mathcal{K}_X$, cf. Theorem 3.12, is $\mathcal{K}_X(J) = \{\mathcal{K}(X^{\otimes m}, X^{\otimes n} J)\}_{n,m\in\mathbb{N}}$.

Corollary 3.22. *Relations* (15) *together with equality $J = \mathcal{J}(0,0)$ establish a one-to-one correspondence between representations of a C^*-correspondence X coisometric on $J \subset J(X)$ and right tensor representations of $\mathcal{K}_X$ coisometric on $\mathcal{J} \subset J(\mathcal{K}_X)$.*

Corollary 3.23 (cf. Prop 2.21 [28])**.** *If $J \subset J(X)$ is an ideal of coisometricity for a representation (π, t) of X, then $J \cap \ker\varphi \subset \ker\pi$. Hence if (π, t) is faithful, then $J \subset (\ker\varphi)^\perp$.*

With the help of the following lemma we will get a version of Theorem 3.21 for the extended representation $\overline{[\pi, t]} = \{\overline{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ introduced in Theorem 3.14. Here P_m stands for the orthogonal projection onto the subspace (18).

Lemma 3.24. *Let (π, t) be a representation of X on a Hilbert space H, and let*

$$J = \{a \in A : \overline{\pi}_{11}(\phi(a)) = \pi(a)\}.$$

Then $J_0 := J \cap J(X)$ is an ideal of coisometricity for (π, t) and

i) *an element $a \in \mathcal{L}(X^{\otimes m}, X^{\otimes n})$ belongs to $\mathcal{L}(X^{\otimes m}, X^{\otimes n} J)$ if and only if $\overline{\pi}_{nm}(a)$ is supported on $P_{m+1}H$, equivalently, $\overline{\pi}_{nm}(a) = \overline{\pi}_{n+1,m+1}(a \otimes 1)$.*

ii) *If $a \otimes 1 \in \mathcal{K}(X^{\otimes m+1}, X^{\otimes n+1})$ and $\overline{\pi}_{nm}(a)$ is supported on $P_{m+1}H$, then $a \in \mathcal{L}(X^{\otimes m}, X^{\otimes n} J_0)$ and in case $J_0 \subset (\ker\phi)^\perp$ (which is always the case when (π, t) is faithful) $a \in \mathcal{K}(X^{\otimes m}, X^{\otimes n} J_0)$.*

Proof. i). Let $a \in \mathcal{L}(X^{\otimes m}, X^{\otimes n} J)$. Then for any $x \in X^{\otimes m}$ there exist $y \in X^{\otimes n}$ and $b \in J$ such that $ax = yb$, and we have

$$\begin{aligned}\overline{\pi}_{nm}(a)\pi_{m,0}(x) &= \pi_{n,0}(ax) = \pi_{n,0}(yb) = \pi_{n,0}(y)\pi(b) = \pi_{n,0}(y)\overline{\pi}_{11}(\phi(b)) \\ &= \overline{\pi}_{n+1,1}(y \otimes 1 \cdot \phi(b)) = \overline{\pi}_{n+1,1}(yb \otimes 1) = \overline{\pi}_{n+1,1}(ax \otimes 1) \\ &= \overline{\pi}_{n+1,m+1}(a \otimes 1)\pi_{m,0}(x).\end{aligned}$$

Thus $\overline{\pi}_{nm}(a) = \overline{\pi}_{n+1,m+1}(a \otimes 1)$ which by Proposition 3.8 iii) is equivalent to $\overline{\pi}_{nm}(a)$ being supported on $P_{m+1}H$.

Conversely, assume that a is such that $\overline{\pi}_{nm}(a)$ is supported on $P_{m+1}H$ (equivalently $\overline{\pi}_{nm}(a) = \overline{\pi}_{n+1,m+1}(a \otimes 1)$). Multiplying $\overline{\pi}_{nm}(a)$ by $\pi_{0,n}(\flat(x))$, $x \in X^{\otimes n}$, from the left and by $\pi_{m,0}(y)$, $y \in X^{\otimes m}$, from the right we get

$$\pi_{0,n}(\flat(x))\overline{\pi}_{nm}(a)\pi_{m,0}(y) = \pi(\flat(x) \cdot a \cdot y) = \pi(\langle x, ay\rangle_A).$$

Analogously for $\overline{\pi}_{n+1,m+1}(a\otimes 1)$ we have

$$\begin{aligned}\pi_{0,n}(\flat(x))\overline{\pi}_{n+1,m+1}(a\otimes 1)\pi_{m,0}(y) &= \overline{\pi}_{11}(\flat(x)\otimes 1\cdot(a\otimes 1)\cdot y\otimes 1)\\ &= \overline{\pi}_{11}(\langle x, ay\rangle\otimes 1) = \overline{\pi}_{11}(\phi(\langle x, ay\rangle_A)).\end{aligned}$$

This implies that $\pi(\langle x, ay\rangle_A) = \overline{\pi}_{11}(\phi(\langle x, ay\rangle_A)$ and hence $\langle x, ay\rangle_A \in J$. By arbitrariness of x and y we conclude that $a \in \mathcal{L}(X^{\otimes m}, X^{\otimes n}J)$.
ii). If $a\otimes 1 \in \mathcal{K}(X^{\otimes m+1}, X^{\otimes n+1})$ and $\overline{\pi}_{nm}(a)$ is supported on $P_{m+1}H$, then the argument form the proof of i) shows that $\pi(\langle x, ay\rangle_A) = \overline{\pi}_{11}(\phi(\langle x, ay\rangle_A))$, and similarly as in the proof of [10, Lemma 4.2 *ii*)], one sees that $\phi(\langle x, ay\rangle_A) \in \mathcal{K}(X)$. Hence we deduce that $\langle x, ay\rangle_A \in J\cap J(X)$ and as a consequence $a \in \mathcal{L}(X^{\otimes m}, X^{\otimes n}J_0)$. If additionally $J_0 \subset (\ker\phi)^{\perp}$, then $a \in \mathcal{K}(X^{\otimes m}, X^{\otimes n}J_0)$ by Lemma 1.8 *ii*). ■

Theorem 3.25. *If (π, t) is a representation of X on a Hilbert space H and*

$$J = \{a \in A : \overline{\pi}_{11}(\phi(a)) = \pi(a)\},$$

then the ideal of coisometricity for the right tensor representation $\overline{[\pi, t]}$ of $\mathcal{T}_X$ is $\mathcal{T}_X(J)$. In particular, if (π, t) is faithful, then J contained in $(\ker\phi)^{\perp}$.

Proof. Apply Proposition 3.8 iii) and Lemma 3.24 i). ■

Example 3.26. Let $\mathcal{T}_\alpha = \{\alpha^m(1)A\alpha^n(1)\}_{n,m\in\mathbb{N}}$ be a right tensor C^*-category associated with an endomorphism $\alpha : A \to A$ of a unital C^*-algebra A, cf. Examples 2.19, 3.4. If (π, U, B) is a covariant representations of α and $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ is the associated right tensor representation of $\mathcal{T}_\alpha$, see Example 3.15, then denoting by $\mathcal{J}$ the ideal of coisometricity for $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ we have

$$\mathcal{J}(m,n) = \alpha^m(1)J\alpha^n(1) \quad \text{where} \quad J = \{a \in A : U^*U\pi(a) = \pi(a)\},$$

cf. [24], [25], [10, Ex. 1.6]. Thus in terminology of [24] the triple (π, U, B) is called a *covariant representation* (π, U, B) *associated with the ideal* J, and we have a one-to-one correspondence between right tensor representations of $\mathcal{T}_\alpha$ with the ideal of coisometricity $\mathcal{J} = \{\alpha^m(1)J\alpha^n(1)\}_{n,m\in\mathbb{N}}$ and covariant representations of α associated with $J = \mathcal{J}(0,0)$. Furthermore, if there exists a complete transfer operator $\mathcal{L}$ for α, cf. Example 3.4, and $J = (\ker\alpha)^{\perp}$ (equivalently $\mathcal{J} = (\ker\otimes 1)^{\perp}$), then

$$U\pi(a)U^* = \pi(\alpha(a)), \qquad U^*\pi(a)U = \pi(\mathcal{L}(a)),$$

that is (π, U, B) is a covariant representation in the sense of [4], [23], [3], cf. [25].
In Example 3.15 we have defined the notion of a covariant representation (π, U, H) of a general partial morphism $\varphi : A \to M(A_0)$ on a Hilbert space H. Every such representation give rise to the right tensor representation $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ of the ideal $\mathcal{K}_\varphi = \{\mathcal{K}(X_\varphi^{\otimes m}, X_\varphi^{\otimes n})\}_{n,m\in\mathbb{N}}$ in $\mathcal{T}_\varphi$, which in turn extends to the right tensor representation $\{\overline{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ of $\mathcal{T}_\varphi$. The ideals of coisometricity $\mathcal{J}$, $\mathcal{J}_0$ for $\{\overline{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ and $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$, are established by the ideals

$$J = \{a \in A : U^*U\pi(a) = \pi(a)\} \quad \text{and} \quad J_0 = J\cap\varphi^{-1}(A_0),$$

respectively. One may see that, if φ arises from partial automorphism, then (π, U, H) is a covariant representation in the sense of [9] iff $J_0 = (\ker\varphi)^{\perp}\cap\varphi^{-1}(A_0)$ (equivalently $\mathcal{J}_0 = (\ker\otimes 1)^{\perp}\cap J(\mathcal{K}_\varphi)$).

Example 3.27. As in Example 3.16, let $\mathcal{T}_E$ be right tensor C^*-precategory of a directed graph $E = (E^0, E^1, r, s)$, $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ a right tensor representation of $\mathcal{K}_E = \{\mathcal{K}(X_E^{\otimes m}, X_E^{\otimes n})\}_{n,m\in\mathbb{N}}$ and $\{p_v : v \in E^0\}$, $\{s_e : e \in E^1\}$ the corresponding

Toeplitz-Cuntz-Krieger E-family. Then the ideal of coisometricity $\mathcal{J}$ for $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ is established via (11) by a set of vertices $V \subset E^0$ where

$$p_v = \sum_{s(e)=v} s_e s_e^*, \qquad \text{if and only if } v \in V.$$

Authors of [29] called an E-family satisfying $p_v = \sum_{s(e)=v} s_e s_e^*$, for $v \in V$, a *Cuntz-Krieger (E,V)-family.* Thus we have a one-to-one correspondence between Cuntz-Krieger (E,V)-families and right tensor representations of $\mathcal{K}_E$ coisometric on the ideal $\mathcal{J}$ corresponding to V. In particular,

$$\mathcal{J} = J(\mathcal{K}_E) \cap (\ker \otimes 1)^{\perp} \quad \text{ iff } \quad V = \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\},$$

and if these equivalent conditions hold, the corresponding (E,V)-family is called a *Cuntz-Krieger E-family* [29], [6], [7].

4. C^*-algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$

In this section we present a construction of the title object of the paper - the C^*-algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$. Before approaching this task, we formulate a universal definition of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ and briefly discuss its relations with relative Cuntz-Pimsner algebras, various crossed products and graph C^*-alg ebras.

4.1. Characterization and examples of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$.

One of the main goals of this section is the proof of the following

Theorem 4.1 (**Characterization of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$**)**.** *For every ideal $\mathcal{K}$ in a right tensor C^*-precategory $\mathcal{T}$ and every ideal $\mathcal{J}$ in $J(\mathcal{K})$ there exists a pair $(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}),i)$ consisting of a C^*-algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ and a right tensor representation $i=\{i_{(n,m)}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ coisometric on $\mathcal{J}$, such that*

1) *the C^*-algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ is generated by the image of the representation $i=\{i_{(n,m)}\}_{n,m\in\mathbb{N}}$, i.e.*
$$\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}) = C^*(\{i_{(n,m)}(\mathcal{K}(m,n))\}_{n,m\in\mathbb{N}}).$$
2) *Every right tensor representation $\pi=\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ coisometric on $\mathcal{J}$ integrates to a representation Ψ_π of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ given by*
$$\Psi_\pi(i_{(n,m)}(a)) = \pi_{nm}(a), \qquad a \in \mathcal{K}(m,n),\ n,,m \in \mathbb{N}.$$

Once we prove this statement the standard argumentation leads us to

Corollary 4.2. *Let $(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}),i)$ be the pair from the thesis of Theorem 4.1.*

i) *The pair $(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}),i)$ is uniquely determined in the sense that if (C,j) is any other pair consisting of a C^*-algebra C and a right tensor representation $j=\{j_{(n,m)}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ in C coisometric on $\mathcal{J}$, that satisfies conditions 1), 2) from Theorem 4.1, then the mappings*
$$i_{(n,m)}(a) \longmapsto j_{(n,m)}(a), \qquad a \in \mathcal{K}(m,n),$$
give rise to the canonical isomorphism $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}) \cong C$.
ii) *There is a canonical circle action $\gamma : S^1 \to Aut(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}))$ by automorphisms of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ given by*
$$\gamma_z(i_{(n,m)}(a)) = z^{m-n} i_{(n,m)}(a), \qquad a \in \mathcal{K}(m,n),\ z \in S^1.$$

Proof. i). By the universality of pairs $(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}),i)$ and (C,j) the mappings $i_{(n,m)}(a)\longmapsto j_{(n,m)}(a)$ and $j_{(n,m)}(a)\longmapsto i_{(n,m)}(a)$, $a\in\mathcal{K}(m,n)$, extend to homomorphisms which are each others inverses.
ii). Fix $z\in S^1$ and consider the family $j=\{j_{(n,m)}\}_{n,m\in\mathbb{N}}$ of mappings given by $j_{(n,m)}(a)=z^{m-n}i_{(n,m)}(a)$, $a\in\mathcal{K}(m,n)$. Clearly, j is a right tensor representation of $\mathcal{K}$ coisometric on $\mathcal{J}$ and the pair $(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}),j)$ satisfy conditions 1), 2) from Theorem 4.1. Thus applying item i) to $(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}),j)$ we get an automorphism γ_z of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ where

$$i_{(n,m)}(a)\overset{\gamma_z}{\longmapsto} j_{(n,m)}(a)=z^{m-n}i_{(n,m)}(a),\qquad a\in\mathcal{K}(m,n).$$

The family $\{\gamma_z\}_{z\in S^1}$ of automorphisms establish a desired group action γ. ■

Definition 4.3. For any ideals $\mathcal{K}$, $\mathcal{J}$ in a right tensor C^*-precategory $\mathcal{T}$ such that $\mathcal{J}\subset J(\mathcal{K})$, the object $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ from the thesis of Theorem 4.1 will be called a *C^*-algebra of the ideal $\mathcal{K}$ in the right tensor C^*-category $\mathcal{T}$ relative to the ideal $\mathcal{J}$.*

Remark 4.4. With analogy to C^*-algebras arising from C^*-correspondences, see paragraph 4.5 below, it is reasonable to call the algebras

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K},\{0\}),\qquad \mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}(\mathcal{K})\cap(\ker\otimes 1)^{\perp}),\qquad \mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}(\mathcal{K}))$$

a *Toeplitz algebra* of $\mathcal{K}$, *(Katsura's) C^*-algebra of the ideal* $\mathcal{K}$, and a *Cuntz-Pimsner algebra* of $\mathcal{K}$, respectively.

In the event $\mathcal{T}$ is a right tensor C^*-category, the Doplicher-Roberts algebra $\mathcal{DR}(\mathcal{T})$ together with the family of natural maps $\pi_{n+k,n}:\mathcal{T}(n,n+k)\to\varinjlim\mathcal{T}(r,r+k)$ (cf. the definition of $\mathcal{DR}(\mathcal{T})$ on page 34 or [12, page 179]) is naturally identified with the algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{T},\mathcal{T})$. We postpone a relevant discussion until section 5, and now present a survey of examples of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ associated with C^*-correspondences.

4.5. *Relative Cuntz-Pimsner algebras.* Let $\mathcal{T}=\mathcal{T}_X$ be the right tensor C^*-precategory of a C^*-correspondence X and let $\mathcal{J}$ be an ideal in $J(\mathcal{K}_X)$. We know that $\mathcal{J}=\mathcal{K}_X(J)=\{\mathcal{K}(X^{\otimes n},X^{\otimes m}J)\}_{n,m\in\mathbb{N}}$ where $J=\mathcal{J}(0,0)\subset J(X)$, cf. Proposition 2.16. In view of Corollary 3.22 and [10, Prop. 1.3] the algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_X,\mathcal{J})$ coincides with the *relative Cuntz-Pimsner algebra* of Muhly and Solel [28]:

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}_X,\mathcal{J})=\mathcal{O}(J,X).$$

In particular, $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_X,\{0\})=\mathcal{O}(\{0\},X)$ is called a *Toeplitz algebra* of X. In case ϕ is injective $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_X,J(\mathcal{K}_X))=\mathcal{O}(J(X),X)$ is the algebra originally introduced by Pimsner, and in the general case the algebra

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}_X,J(\mathcal{K}_X)\cap(\ker\otimes 1)^{\perp})=\mathcal{O}(J(X)\cap(\ker\phi)^{\perp},X),$$

is the *C^*-algebra $\mathcal{O}_X$ of the correspondence* X investigated and popularized by T. Katsura [16], [17], [18].

As subclasses of the above algebras we get

4.6. *Relative graph algebras.* Let $\mathcal{T}=\mathcal{T}_E$ be a right tensor C^*-precategory of a directed graph E and let $\mathcal{K}$ be the ideal in $\mathcal{T}$ spanned by the matrix units (10). Then any ideal $\mathcal{J}$ in $\mathcal{K}$ is determined by a set of vertices $V\subset E^0$, see (11). In view of Example 3.27, $V\subset\{v\in E^0: 0<s^{-1}(v)<\infty\}$ iff $\mathcal{J}\subset J(\mathcal{K})\cap(\ker\otimes 1)^{\perp}$ and in this event we have

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})=C^*(E,V)$$

where $C^*(E,V)$ is the *relative graph algebra* introduced in [29, Def. 3.5]. In particular, $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\{0\}) = C^*(E,\emptyset)$ is the *Toeplitz algebra of* E as defined in [11], and

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}, J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}) = C^*(E, \{v \in E^0 : 0 < s^{-1}(v) < \infty\}) = C^*(E)$$

where $C^*(E)$ is the *graph algebra*, cf. [29], [6], [7]. We extend [29, Def. 3.5] and put $C^*(E,V) := \mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ for any $V \subset \{v \in E^0 : s^{-1}(v) < \infty\}$ (that is for any $\mathcal{J} \subset J(\mathcal{K})$).

4.7. *Crossed products by partial morphisms.* Let $\mathcal{T} = \mathcal{T}_\varphi$ be the right tensor C^*-precategory associated with a partial morphism φ. For the ideal $\mathcal{K}$ described by (9) we have

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}, J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}) = A \rtimes_{\varphi} \mathbb{N}$$

where $A \rtimes_\varphi \mathbb{N}$ is the *crossed product by partial morphism* defined in [16]. In particular, if φ arises from partial automorphism, this algebra coincides with the Exel's crossed product by partial automorphism [9].

4.8. *Partial isometric crossed products by endomorphisms.* Let $\mathcal{T}$ be the right tensor C^*-category $\mathcal{T}_\alpha = \{\alpha^m(1)A\alpha^n(1)\}_{m,n\in\mathbb{N}}$ associated with an endomorphism $\alpha : A \to A$ of a unital C^*-algebra A, cf. Example 3.4. Then every ideal $\mathcal{J}$ in $\mathcal{T}$ is of the form $\mathcal{J} = \{\alpha^m(1)J\alpha^n(1)\}_{m,n\in\mathbb{N}}$ for an ideal J in A, and

$$\mathcal{O}_{\mathcal{T}}(\mathcal{T},\mathcal{J}) = C^*(A,\alpha;J)$$

where $C^*(A,\alpha;J)$ is the *relative crossed product* considered in [24], [25], cf. Example 3.26. Thereby, see [25], for $J = A$, $J = (\ker\alpha)^{\perp}$ or $J = \{0\}$ one arrives at partial-isometric crossed product from [2], [19] or [27], respectively. Furthermore, if there exists a complete transfer operator $\mathcal{L}$ for α, then

$$C^*(A,\alpha;(\ker\alpha)^{\perp}) = \mathcal{O}_{\mathcal{T}}(\mathcal{T}, (\ker \otimes 1)^{\perp})$$

is the crossed product investigated in [23], [3]. A distinctive property of C^*-algebras of this type is that they are generated by a homomorphic image of A and a partial isometry.

The last two paragraphs indicate that it is reasonable to make the following definition which embraces all the crossed products mentioned above.

Definition 4.9. Let $\varphi : A \to M(A_0)$ be a partial morphism and let J be an ideal in $\varphi^{-1}(A_0)$. We define the *relative crossed product* $C^*(\varphi; J)$ of φ relative to J to be the C^*-algebra $\mathcal{O}(J, X_\varphi)$.

4.2. **Construction of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$.** Let $\mathcal{K}$ and $\mathcal{J} \subset \mathcal{K}$ be ideals in a right tensor C^*-precategory $\mathcal{T}$. An explicit construction of the C^*-algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ will consist of two steps. Firstly, we describe a matrix calculus that yields a purely algebraic structure - a *-algebra $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$. Secondly, we use $\mathcal{J}$ to define a seminorm $\|\cdot\|_{\mathcal{J}}$ on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ such that completing the quotient space $\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}$ we arrive at $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$.

4.2.1. *An algebraic framework $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ for $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$.* We denote by $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ the set of infinite matrices $\{a_{nm}\}_{n,m\in\mathbb{N}}$ such that

$$a_{nm} \in \mathcal{K}(m,n), \qquad n,\, m \in \mathbb{N},$$

and there is at most finite number of elements a_{nm} which are non-zero. We define the addition, multiplication by scalars, and involution on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ in a quite natural

manner: for $a = \{a_{nm}\}_{n,m\in\mathbb{N}}$ and $b = \{b_{nm}\}_{n,m\in\mathbb{N}}$ we put

$$(a+b)_{nm} := a_{nm} + b_{nm}, \tag{23}$$

$$(\lambda a)_{nm} := \lambda a_{nm} \tag{24}$$

$$(a^*)_{nm} := a^*_{mn}. \tag{25}$$

A convolution multiplication "$\star$" on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ is defined in a more sophisticated way. Namely, we set

$$a \star b = a \cdot \sum_{k=0}^{\infty} \Lambda^k(b) + \sum_{k=1}^{\infty} \Lambda^k(a) \cdot b \tag{26}$$

where "$\cdot$" is the standard multiplication of matrices and mapping $\Lambda : \mathcal{M}_{\mathcal{T}}(\mathcal{K}) \to \mathcal{M}_{\mathcal{T}}(\mathcal{K})$ is defined to act as follows

$$\Lambda(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{00}\otimes 1 & a_{01}\otimes 1 & a_{02}\otimes 1 & \cdots \\ 0 & a_{10}\otimes 1 & a_{11}\otimes 1 & a_{12}\otimes 1 & \cdots \\ 0 & a_{20}\otimes 1 & a_{21}\otimes 1 & a_{22}\otimes 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{27}$$

that is $\Lambda(a)_{nm} = a_{n-1,m-1} \otimes 1$, for $n, m > 1$, and $\Lambda(a)_{nm} = 0$ otherwise. Note that even though the entries of $\Lambda(a)$ need not belong to $\mathcal{K}$, entries of $a \star b$ are in $\mathcal{K}$ (because $\mathcal{K}$ is an ideal in $\mathcal{T}$) and hence $a \star b$ is well defined.

We denote by $i^{\mathcal{M}}_{(n,m)} : \mathcal{T}(m,n) \to \mathcal{M}_{\mathcal{T}}(\mathcal{K})$, $n, m \in \mathbb{N}$, the natural embeddings, that is $i^{\mathcal{M}}_{(n,m)}(a)$ stands for a matrix $\{a_{ij}\}_{i,j\in\mathbb{N}}$ satisfying $a_{ij} = \delta_{in}\delta_{jm}a$ where δ_{kl} is the Kronecker symbol.

Proposition 4.10. *The set $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ with operations* (23), (24), (25), (26) *becomes an algebra with involution, and the family $\{i^{\mathcal{M}}_{(n,m)}\}_{n,m\in\mathbb{N}}$ forms a right tensor representation of $\mathcal{K}$ in $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$.*

Proof. We only show the associativity of the multiplication (26), as the rest is a completely routine computation. For that purpose note that Λ preserves the standard matrix multiplication and thus we have

$$\begin{aligned}
a\star(b\star c) &= a \star \Big(b \cdot \sum_{j=0}^{\infty} \Lambda^j(c) + \sum_{j=1}^{\infty} \Lambda^j(b)\cdot c\Big) \\
&= a\sum_{k=0}^{\infty}\Lambda^k\Big(b\sum_{j=0}^{\infty}\Lambda^j(c) + \sum_{j=1}^{\infty}\Lambda^j(b)c\Big) + \sum_{k=1}^{\infty}\Lambda^k(a)\Big(b\sum_{j=0}^{\infty}\Lambda^j(c) + \sum_{j=1}^{\infty}\Lambda^j(b)c\Big) \\
&= \sum_{k,j=0}^{\infty} a\Lambda^k(b)\Lambda^j(c) + \sum_{k=1,j=0}^{\infty} \Lambda^k(a)b\cdot\Lambda^j(c) + \sum_{k,j=1}^{\infty}\Lambda^k(a)\Lambda^j(b)c \\
&= \Big(a\sum_{k=0}^{\infty}\Lambda^k(b) + \sum_{k=1}^{\infty}\Lambda^k(a)b\Big)\sum_{j=0}^{\infty}\Lambda^j(c) + \sum_{j=1}^{\infty}\Lambda^j\Big(a\sum_{k=0}^{\infty}\Lambda^k(b) + \sum_{k=1}^{\infty}\Lambda^k(a)b\Big)c \\
&= \Big(a\cdot\sum_{k=0}^{\infty}\Lambda^k(b) + \sum_{k=1}^{\infty}\Lambda^k(a)\cdot b\Big)\star c = (a\star b)\star c.
\end{aligned}$$

■

Theorem 4.11. *If $\pi = \{\pi_{nm}\}_{n,m\in\mathbb{N}}$ is a right tensor representation of an ideal $\mathcal{K}$ in a right C^*-category $\mathcal{T}$, then the formula*

$$\Psi_\pi(\{a_{nm}\}_{n,m\in\mathbb{N}}) = \sum_{n,m=0}^{\infty} \pi_{nm}(a_{nm}) \tag{28}$$

defines a *-representation Ψ_π of $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$.

Proof. We shall only check the multiplicativity of Ψ_π as the rest is straightforward. We fix two matrices $a = \{a_{m,n}\}_{m,n\in\mathbb{N}}, b = \{b_{m,n}\}_{m,n\in\mathbb{N}} \in \mathcal{M}_{\mathcal{T}}(\mathcal{K})$ and examine the product

$$\begin{aligned}\Psi_\pi(a)\Psi_\pi(b) &= \sum_{p,r,s,t\in\mathbb{N}} \pi_{p,r}(a_{pr})\pi_{s,t}(b_{st})\\ &= \sum_{\substack{p,r,s,t\in\mathbb{N}\\ s\leq r}} \pi_{p,r}(a_{pr})\pi_{s,t}(b_{st}) + \sum_{\substack{p,r,s,t\in\mathbb{N}\\ r<s}} \pi_{p,r}(a_{pr})\pi_{s,t}(b_{st}).\end{aligned}$$

1) If $s \leq r$, then

$$\pi_{p,r}(a_{pr})\pi_{s,t}(b_{st}) = \pi_{p,t+r-s}(a_{pr}(b_{st}\otimes 1^{r-s}))$$

and by the change of indexes: $j := r-s$, $i := r$, $m := p$, $n := t+r-s$,

$$\pi_{p,r}(a_{pr})\pi_{s,t}(a_{st}) = \pi_{m,n}(a_{mi}(b_{i-j,n-j}\otimes 1^{j})).$$

Thus

$$\sum_{\substack{p,r,s,t\in\mathbb{N}\\ s\leq r}} \pi_{p,r}(a_{pr})\pi_{s,t}(b_{st}) = \sum_{j=0}^{\infty}\sum_{i=j}^{\infty} \pi_{m,n}(a_{mi}(b_{i-j,n-j}\otimes 1^{j})) = \pi_{mn}\big((a\cdot \sum_{j=0}^{\infty}\Lambda^j(b))_{mn}\big).$$

2) If $r < s$, then

$$\pi_{p,r}(a_{pr})\pi_{s,t}(b_{st}) = \pi_{p+s-r,t}((a_{pr}\otimes 1^{s-r})b_{st})$$

and by the change of indexes: $j := s-r$, $i := s$, $m := p+s-r$, $n := t$

$$\pi_{p,r}(a_{pr})\pi_{s,t}(b_{st}) = \pi_{mn}((a_{m-j,i-j}\otimes 1^{j})b_{in}).$$

Thus

$$\sum_{\substack{s,r\in\mathbb{N}\\ r<s}} \pi_{p,r}(a_{pr})\pi_{s,t}(b_{st}) = \sum_{j=1}^{\infty}\sum_{i=j}^{\infty} \pi_{m,n}((a_{m-j,i-j}\otimes 1^{j})b_{in}) = \pi_{mn}\big((\sum_{j=0}^{\infty}\Lambda^j(a)\cdot b)_{mn}\big).$$

Finally, making use of the formulas obtained in 1) and 2) we have

$$\begin{aligned}\Psi_\pi(a)\Psi_\pi(b) &= \pi_{mn}\Big(\big(a\cdot\sum_{j=0}^{\infty}\Lambda^j(b)\big)_{mn}\Big) + \pi_{mn}\Big(\big(\sum_{j=0}^{\infty}\Lambda^j(a)\cdot b\big)_{mn}\Big)\\ &= \sum_{m,n\in\mathbb{N}} \pi_{mn}((a\star b)_{m,n}) = \Psi_\pi(a\star b)\end{aligned}$$

and the proof is complete. ■

4.2.2. *Grading and seminorms $\|\cdot\|_{\mathcal{J}}$ in $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$.* We define a seminorm $\|\cdot\|_{\mathcal{J}}$ in $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ using a natural grading of $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$. We also take an opportunity to show that this seminorm can be defined within the inductive limit frames based on Doplicher-Roberts method. For that purpose we put

$$\mathcal{M}(r+k,r) := \{\{a_{nm}\}_{m,n\in\mathbb{N}} \in \mathcal{M}_{\mathcal{T}}(\mathcal{K}) : \ a_{nm}\neq 0 \Longrightarrow \ m-n=k, \ n\leq r\}.$$

Hence an element $a\in\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ is in $\mathcal{M}(r+k,r)$ iff it is of the form

$$\begin{pmatrix} & a_{0,k} & & 0 \\ & & \ddots & \\ & & & a_{r,r+k} \\ 0 & & & \end{pmatrix}, \text{ if } \ k\geq 0, \text{ or } \qquad \begin{pmatrix} & & & 0 \\ a_{-k,0} & & & \\ & \ddots & & \\ & & a_{r,r+k} & \\ 0 & & & \end{pmatrix}, \text{ if } \ k\leq 0.$$

For every $k\in\mathbb{Z}$ we get an increasing family $\{\mathcal{M}(r,r+k)\}_{r\in\mathbb{N},\, r+k\geq 0}$ of subspaces of $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$. If we put

$$\mathcal{M}^{(k)}_{\mathcal{T}}(\mathcal{K}) = \bigcup_{\substack{r\in\mathbb{N},\\ r+k\geq 0}} \mathcal{M}(r+k,r), \qquad k\in\mathbb{Z},$$

we get a family of subspaces that defines a $\mathbb{Z}$-grading on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$. Namely, we may present $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ as the following direct sum

$$\mathcal{M}_{\mathcal{T}}(\mathcal{K}) = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}^{(k)}_{\mathcal{T}}(\mathcal{K}).$$

The first step in defining $\|\cdot\|_{\mathcal{J}}$ is to equip the family $\{\mathcal{M}(m,n)\}_{m,n\in\mathbb{N}}$ with the structure of a right tensor C^*-precategory. We recall that $q_{\mathcal{J}}$ denotes the quotient map from $\mathcal{T}$ onto $\mathcal{T}/\mathcal{J}$, cf. Proposition 2.10.

Theorem 4.12 (Construction of the right tensor C^*-precategory $\mathcal{K}_{\mathcal{J}}$)**.** *For every $r\in\mathbb{N}$, $k\in\mathbb{Z}$, $r+k\geq 0$, the formula*

$$\|a\|^{\mathcal{J}}_{r+k,r} := \max\left\{ \max_{s=0,\dots,r-1} \Big\{\Big\|q_{\mathcal{J}}\Big(\sum_{\substack{i=0,\\ i+k\geq 0}}^{s} a_{i,i+k}\otimes 1^{s-i}\Big)\Big\|\Big\},\ \Big\|\sum_{\substack{i=0,\\ i+k\geq 0}}^{r} (a_{i,i+k}\otimes 1^{r-i})\Big\| \right\},$$

defines a seminorm on $\mathcal{M}(r+k,r)$, such that the family of quotients spaces

$$\mathcal{K}_{\mathcal{J}} := \{\mathcal{M}(m,n)/\|\cdot\|^{\mathcal{J}}_{m,n}\}_{m,n\in\mathbb{N}}$$

form a right tensor C^*-precategory with a right tensoring $\otimes_{\mathcal{J}}1$ induced by the inclusions $\mathcal{M}(m,n)\subset\mathcal{M}(m+1,n+1)$, $m,n\in\mathbb{N}$. Moreover the right tensoring $\otimes_{\mathcal{J}}1$ is faithful iff $\mathcal{J}\subset(\ker\otimes 1)^{\perp}$.

Proof. Using (25), (26) one readily checks the relations

$$\mathcal{M}(m,n)\star\mathcal{M}(l,m)\subset\mathcal{M}(l,n), \qquad \mathcal{M}(m,n)^* = \mathcal{M}(n,m), \qquad l,m,n\in\mathbb{N},$$

which imply that $\{\mathcal{M}(m,n)\}_{m,n\in\mathbb{N}}$ is a *-precategory. It is clear that functions $\|\cdot\|^{\mathcal{J}}_{r+k,r}$ are seminorms on $\mathcal{M}(r+k,r)$. To see that they satisfy the C^*-equality let $a\in\mathcal{M}(r+k,r)$ and $s\in\mathbb{N}$. A direct computation shows that

$$\sum_{\substack{i=0,\\ i+k\geq 0}}^{s} (a^*\star a)_{i,i}\otimes 1^{s-i} = \begin{cases} \left(\sum_{i=0}^{s-k} a_{i,i+k}\otimes 1^{s-k-i}\right)^* \left(\sum_{j=0}^{s-k} a_{j,j+k}\otimes 1^{s-k-j}\right) & k\geq 0 \\ \left(\sum_{i=-k}^{s} a_{i,i+k}\otimes 1^{s-i}\right)^* \left(\sum_{j=-k}^{s} a_{j,j+k}\otimes 1^{s-j}\right) & k<0 \end{cases}.$$

Since $q_{\mathcal{J}}$ preserves the operations in $\mathcal{T}$ we thus have

$$\max_{s=1,\dots,r+k-1} \|q_J\big(\sum_{\substack{i=0,\\ i+k\geq 0}}^{s} (a^*\star a)_{i,i}\otimes 1^{s-i}\big)\| = \max_{s=1,\dots,r-1} \|q_{\mathcal{J}}\Big(\sum_{\substack{i=0,\\ i+k\geq 0}}^{s} a_{i,i+k}\otimes 1^{s-i}\Big)\|^2$$

and

$$\|\sum_{\substack{i=0,\\ i+k\geq 0}}^{r+k} (a^*\star a)_{i,i}\otimes 1^{r+k-i}\big)\| = \|\sum_{\substack{i=0,\\ i+k\geq 0}}^{r} a_{i,i+k}\otimes 1^{r-i}\|^2,$$

that is $\|a^*\star a\|^{\mathcal{J}}_{r+k,r+k} = \big(\|a\|^{\mathcal{J}}_{r+k,r}\big)^2$.
Similar direct computations show that for $a\in\mathcal{M}(m,n)$ and $b\in\mathcal{M}(l,m)$, $l,m,n\in\mathbb{N}$, we have

$$\|a\star b\|^{\mathcal{J}}_{l,n}\leq \|a\|^{\mathcal{J}}_{m,n}\|b\|^{\mathcal{J}}_{l,m}.$$

Viewing the quotient space $\mathcal{M}(m,n)/\|\cdot\|^{\mathcal{J}}_{m,n}$ as quotient of the Banach space $\mathcal{M}(m,n)$ with norm $\|\cdot\|^{\{0\}}_{m,n}$, by the closed subspace $\{a\in\mathcal{M}(m,n):\|\,a\|^{\mathcal{J}}_{m,n}=0\}$, one gets that the space $\mathcal{M}(m,n)/\|\cdot\|^{\mathcal{J}}_{m,n}$ is already complete. Hence the family $\{\mathcal{M}(m,n)/\|\cdot\|^{\mathcal{J}}_{m,n}\}$ forms a C^*-precategory.

We now prove that inclusion $\mathcal{M}(r+k,r)\subset\mathcal{M}(r+k+1,r+1)$ factors through to the mapping

$$\mathcal{M}(r+k,r)/\|\cdot\|^{\mathcal{J}}_{r,r}\xrightarrow{\otimes_{\mathcal{J}}1}\mathcal{M}(r+k+1,r+1)/\|\cdot\|^{\mathcal{J}}_{r+1,r+1}.$$

Let $a\in\mathcal{M}(r+k,r)$ be such that $\|a\|^{\mathcal{J}}_{r+k,r}=0$. Then $\sum^{s}_{\substack{i=0,\\ i+k\geq 0}} a_{i,i+k}\otimes 1^{s-i}\in\mathcal{J}(s+k,s)$, for all $s=0,1,....,r-1$ $(s+k\geq 0)$, and $\sum^{r}_{\substack{i=0,\\ i+k\geq 0}}(a_{i,i+k}\otimes 1^{r-i})=0$. It follows that $\sum^{s}_{\substack{i=0,\\ i+k\geq 0}} a_{i,i+k}\otimes 1^{s-i}\in\mathcal{J}(s+k,s)$, for all $s=0,1,....,r$, and

$$\sum_{\substack{i=0,\\ i+k\geq 0}}^{r+1} a_{i,i+k}\otimes 1^{r+1-i} = \sum_{\substack{i=0,\\ i+k\geq 0}}^{r} a_{i,i+k}\otimes 1^{r+1-i} = (\sum_{\substack{i=0,\\ i+k\geq 0}}^{r} a_{i,i+k}\otimes 1^{r-i})\otimes 1 = 0\otimes 1 = 0.$$

Thus $\|a\|^{\mathcal{J}}_{r+1,r+1}=0$ and the right tensoring $\otimes_{\mathcal{J}}1$ is well defined.
Assume now that $\mathcal{J}\subset(\ker\otimes 1)^{\perp}$ and $a\in\mathcal{M}(r+k,r)$ is such that $\|a\|^{\mathcal{J}}_{r+k+1,r+1}=0$. Then $\sum^{s}_{\substack{i=0,\\ i+k\geq 0}} a_{i,i+k}\otimes 1^{s-i}\in\mathcal{J}(s+k,s)$, for each $s=0,1,....,r$, and

$$\sum_{\substack{i=0,\\ i+k\geq 0}}^{r+1} a_{i,i+k}\otimes 1^{r+1-i} = (\sum_{\substack{i=0,\\ i+k\geq 0}}^{r} a_{i,i+k}\otimes 1^{r-i})\otimes 1 = 0.$$

In particular $\sum^{r}_{\substack{i=0,\\ i+k\geq 0}} a_{i,i+k}\otimes 1^{r-i}\in\ker(\otimes 1)(r+k,r)$ and $\sum^{r}_{\substack{i=0,\\ i+k\geq 0}} a_{i,i+k}\otimes 1^{r-i}\in\mathcal{J}(r+k,r)$. Since $\mathcal{J}\cap\ker(\otimes 1)=\{0\}$, we get $\sum^{r}_{\substack{i=0,\\ i+k\geq 0}} a_{i,i+k}\otimes 1^{r-i}=0$ and thereby $\|a\|^{\mathcal{J}}_{r+k,r}=0$. This proves the injectivity of $\otimes_{\mathcal{J}}1$.
Conversely, if $\mathcal{J}\nsubseteq(\ker\otimes 1)^{\perp}$, then there exists $a\in\mathcal{J}(n,m)\cap\ker\otimes 1(n,m)$ such that $a\neq 0$. Clearly $i_{(m,n)}(a)\in\mathcal{M}(m,n)$, $\|i_{(m,n)}(a)\|^{\mathcal{J}}_{n,m}\neq 0$ and $\|i_{(m,n)}(a)\|^{\mathcal{J}}_{n+1,m+1}=0$, that is $\otimes_{\mathcal{J}}1$ is not injective. ∎

The foregoing statement allow us to apply the "Doplicher-Roberts" method of constructing C^*-algebras from right tensor C^*-categories presented in [12]. Namely,

for $k\in\mathbb{Z}$, we let $\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{J},\mathcal{K}) := \varinjlim \mathcal{K}_{\mathcal{J}}(r,r+k)$ to be the Banach space inductive limit of the inductive sequence

$$\mathcal{K}_{\mathcal{J}}(r+k,r) \stackrel{\otimes_{\mathcal{J}}1}{\longrightarrow} \mathcal{K}_{\mathcal{J}}(r+k+1,r+1) \stackrel{\otimes_{\mathcal{J}}1}{\longrightarrow} \mathcal{K}_{\mathcal{J}}(r+k+2,r+2) \stackrel{\otimes_{\mathcal{J}}1}{\longrightarrow} ...$$

defined for $r\in\mathbb{N}$ such that $k+r\geq 0$. The algebraic direct sum $\bigoplus_{k\in\mathbb{Z}}\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K},\mathcal{J})$ has a natural structure of $\mathbb{Z}$-graded *-algebra and we define $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ to be the C^*-algebra obtained by completing $\bigoplus_{k\in\mathbb{Z}}\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K},\mathcal{J})$ in the unique C^*-norm for which the automorphic action defined by the grading is isometric, see [12, Thm. 4.2]. Alternatively we may consider $\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K},\mathcal{J})$ as the completion of the quotient subspace $\mathcal{M}_{\mathcal{T}}^{(k)}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}$ where $\|\cdot\|_{\mathcal{J}}$ is a seminorm described below, and thus obtain perhaps a more direct way of constructing $\mathcal{O}_{\mathcal{T}}(\mathcal{J},\mathcal{K})$.

Proposition 4.13 ("Explicit" construction of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$)**.** *The formula*

$$\|a\|_{\mathcal{J}} = \sum_{k\in\mathbb{Z}} \lim_{r\to\infty} \max\left\{ \max_{s=0,\dots,r-1}\big\{\|q_{\mathcal{J}}\Big(\sum_{\substack{i=0,\\ i+k\geq 0}}^{s} a_{i,i+k}\otimes 1^{s-i}\Big)\|\big\},\ \|\sum_{\substack{i=0,\\ i+k\geq 0}}^{r}(a_{i,i+k}\otimes 1^{r-i})\| \right\}$$

defines a submultiplicative *-seminorm on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ such that the enveloping C^*-algebra of the quotient *-algebra $\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}$ is naturally isomorphic to $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$.

Proof. It suffices to notice that the norm of the image of an element $a\in\mathcal{M}_{\mathcal{T}}^{(k)}(\mathcal{K})$ in the inductive limit space $\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K},\mathcal{J})$ coincides with the value $\|a\|_{\mathcal{J}}$, that is $\|a\|_{\mathcal{J}} = \lim_{r\to\infty}\|a\|_{r+k,r}^{\mathcal{J}}$. In particular, $\|\cdot\|_{\mathcal{J}}$ is submultiplicative *-seminorm on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ and clearly enveloping C^*-norm on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}$ satisfy the conditions of [12, Thm. 4.2]. ■

The only thing left for us now to prove is that $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ satisfies the universal conditions presented in Theorem 4.1.

Theorem 4.14 (Universality of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$)**.** *Let $\mathcal{J}$ be an ideal in $\mathcal{T}$ such that $\mathcal{J}\subset J(\mathcal{K})$. The right tensor representation described in Proposition 4.10 factors thorough to the right tensor representation $i=\{i_{(n,m)}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ which is coisometric on $\mathcal{J}$. The pair $(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}),i)$ satisfies the following conditions*

1) *the C^*-algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ is generated by the image of the representation $\{i_{(n,m)}\}_{n,m\in\mathbb{N}}$, i.e.*
$$\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}) = C^*(\{i_{(n,m)}(\mathcal{K}(n,m))\}_{n,m\in\mathbb{N}}).$$
2) *for every right tensor representation $\pi=\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ which is coisometric on $\mathcal{J}$ the representation Ψ_π described in Theorem 4.11 factors through to the representation of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ such that*
$$\Psi_\pi(i_{(n,m)}(a)) = \pi_{nm}(a), \qquad a\in\mathcal{K}(m,n).$$

Proof. It is obvious that $\{i_{(n,m)}^{\mathcal{M}}\}_{n,m\in\mathbb{N}}$ factors thorough to the right tensor representation $i\{i_{(n,m)}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$. To see that i is coisometric on $\mathcal{J}$ take $a\in J(\mathcal{K})(n,m)$, $n,m\in\mathbb{N}$, and notice that

$$\begin{aligned} i_{(n,m)}(a) = i_{(n+1,m+1)}(a\otimes 1) \quad &\Longleftrightarrow \quad \|i_{(n,m)}^{\mathcal{M}}(a) - i_{(n+1,m+1)}^{\mathcal{M}}(a\otimes 1)\|_{\mathcal{J}} = 0 \\ &\Longleftrightarrow \quad a\in\mathcal{J}(n,m) \end{aligned} \tag{29}$$

Item 1) is clear. Let us prove 2). Since $\|\cdot\|_{\mathcal{J}}$ satisfies the C^*-equality it is enough to show that Ψ_π factors through to a representation of the *-algebra $\mathcal{M}_{\mathcal{T}}^{(0)}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}$.

For that purpose let $a \in \mathcal{M}_{\mathcal{T}}^{(0)}(\mathcal{K})$ be such that $\|a\|_{\mathcal{J}} = 0$. Then

$$\sum_{i=0}^{s} a_{ii} \otimes 1^{s-i} \in \mathcal{J}(s,s), \quad s \in \mathbb{N} \qquad \lim_{r\to\infty} \|\sum_{i=0}^{r} a_{ii} \otimes 1^{r-i}\| = 0.$$

Thus, for sufficiently large N we have

$$\begin{aligned} \|\Psi_\pi(a)\| = \|\sum_{i=0}^{N} \pi_{ii}(a_{ii})\| &= \|\pi_{NN}\left(\sum_{i=0}^{N} a_{ii} \otimes 1^{N-i}\right)\| \\ &\leq \|\sum_{i=0}^{N} a_{ii} \otimes 1^{N-i}\| \longrightarrow 0, \qquad \text{as} \qquad N \longrightarrow \infty, \end{aligned}$$

and item 2) follows. ■

Remark 4.15. In the process of construction of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$, until the above statement, we did not require that $\mathcal{J} \subset J(\mathcal{K})$. Actually for any ideal $\mathcal{J}$ in $\mathcal{T}$, Proposition 4.13 defines the C^*-algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$. However, without the condition $\mathcal{J} \subset J(\mathcal{K})$ the relationship between representations of $\mathcal{K}$ and $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ breaks down.

4.3. Immediate corollaries of the construction of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$.

Proposition 4.16 (Norm of elements in spectral subspaces)**.** *Let $(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}), i)$ be the universal pair as in Theorem 4.1. Then for every $k \in \mathbb{Z}$, the elements of the form*

$$a = \sum_{\substack{j=0,\\ j+k\geq 0}}^{r_0} i_{(j,j+k)}(a_{j,j+k}), \qquad r_0 \in \mathbb{N} \tag{30}$$

form a dense subspace of the k-th spectral subspace of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$, and

$$\|a\| = \lim_{r\to\infty} \max\left\{ \max_{s=0,\dots,r-1} \{\|q_{\mathcal{J}}\Big(\sum_{\substack{i=0,\\ i+k\geq 0}}^{s} a_{i,i+k} \otimes 1^{s-i}\Big)\|\}, \|\sum_{\substack{i=0,\\ i+k\geq 0}}^{r} (a_{i,i+k} \otimes 1^{r-i})\| \right\}.$$

If $\mathcal{J} \subset (\ker \otimes 1)^\perp$ the above formula reduces to the following one

$$\|a\| = \max\left\{ \max_{s=0,\dots,r_0-1} \{\|q_{\mathcal{J}}\Big(\sum_{\substack{i=0,\\ i+k\geq 0}}^{s} a_{i,i+k} \otimes 1^{s-i}\Big)\|\}, \|\sum_{\substack{i=0,\\ i+k\geq 0}}^{r_0} (a_{i,i+k} \otimes 1^{r_0-i})\| \right\}.$$

Proof. In view of Theorem 4.12, the function $\|\cdot\|_{\mathcal{J}}$ defined in Proposition 4.13 satisfies the C^*-equality: $\|a^*a\|_{\mathcal{J}} = \|a\|^2_{\mathcal{J}}$, $a \in \mathcal{M}_{\mathcal{T}}^{(k)}(\mathcal{K})$, $k \in Z$. In particular, it is a C^*-seminorm on $\mathcal{M}_{\mathcal{T}}^{(0)}(\mathcal{K})$ and hence (by the uniqueness of the C^*-norm) the norm of the element (30) is given by the same formula as the $\|\cdot\|_{\mathcal{J}}$-norm of the corresponding element of $\mathcal{M}_{\mathcal{T}}^{(k)}(\mathcal{K})$. The last part of the statement follows from the second part of Theorem 4.12. ■

Corollary 4.17 (Kernel of universal representation)**.** *The universal representation $i = \{i_{(n,m)}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ is faithful if and only if $\mathcal{J} \subset (\ker \otimes 1)^\perp$, and in general we have*

$$a \in \ker i_{(n,m)} \iff \lim_{k\to\infty} \|a \otimes 1^k\| = 0 \quad \textit{and} \quad a \otimes 1^k \in \mathcal{J} \quad \textit{for all } k \in \mathbb{N}. \tag{31}$$

Corollary 4.18. *The universal representation of a C^*-correspondence X in the C^*-algebra $\mathcal{O}(J,X)$ is faithful if and only if $J \subset (\ker\phi)^\perp$.*

The ideal described by (31) - the kernel of $i = \{i_{(n,m)}\}_{n,m\in\mathbb{N}}$, will play an important role in the sequel, cf. Definition 6.8 and Theorem 6.9. The condition (32) presented below plays an essential role in investigation of faithful representations arising form right tensor representations, cf. Theorem 7.3. In the context of C^*-correspondences condition (32) leads to the notion of a T-pair introduced in [18], see Subsection 7.1.

Proposition 4.19 (Ideals of coisometricity for universal representation)**.** *The ideal of coisometricity for the universal representation $i = \{i_{(n,m)}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ coincides with $\mathcal{J}$. Moreover, we have*

$$\mathcal{J}(n,n) = i_{n,n}^{-1}\left(\sum_{j=1}^{k} i_{n+j,n+j}(\mathcal{K}(n+j,n+j))\right), \qquad n,k\in\mathbb{N}. \tag{32}$$

Proof. The form of the ideal of coisometricity for $\{i_{(n,m)}\}_{n,m\in\mathbb{N}}$ follows from (29). Actually, slightly modifying (29) one gets (32). ■

Corollary 4.20. *Every ideal $\mathcal{J}$ in $J(\mathcal{K})$ is an ideal of coisometricity for certain right tensor representation of $\mathcal{K}$.*

Corollary 4.21. *Let X be a C^*-correspondence. Every ideal J in $J(X)$ is an ideal of coisometricity for a certain representation (π,t) of X.*

Example 4.22. Let $\varphi : A \to M(A_0)$ be a partial morphism. In view of Example 3.26, for every ideal J in $\varphi^{-1}(A_0)$ there exists a covariant representation (π,U,H) such that

$$J = \{a \in A : U^*U\pi(a) = \pi(a)\},$$

consult with analogous results of [24], [25].

Example 4.23. By Example 3.27 for every set of vertices $V \subset E^0$ in a directed graph $E = (E^0,E^1,r,s)$ there exists a Cuntz-Krieger (E,V)-family (which is not an (E,V')-family for any V' bigger than V)

5. $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ vs $\mathcal{DR}(\mathcal{T})$ and algebras with circle actions

The *Doplicher-Roberts algebra $\mathcal{DR}(\mathcal{T})$ of a right tensor C^*-category $\mathcal{T}$* is defined to be the completion of the algebraic direct sum $\bigoplus_{k\in\mathbb{Z}}\mathcal{DR}^{(k)}(\mathcal{T})$ where

$$\mathcal{DR}^{(k)}(\mathcal{T}) := \varinjlim \mathcal{T}(r,r+k)$$

in the unique C^*-norm for which the automorphic action defined by the grading is isometric [12, page 179]. Plainly this construction could be successfully applied to C^*-precategories (or even to ideals in C^*-precategories). Thus we slightly extend existing nomenclature and, for any right tensor C^*-precategory $\mathcal{T}$, call the C^*-algebra $\mathcal{DR}(\mathcal{T})$ defined above a *Doplicher-Roberts algebra of the right tensor C^*-precategory $\mathcal{T}$*.

Proposition 5.1 (Relationship between $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ and Doplicher-Roberts algebras)**.** *The Doplicher-Roberts algebra $\mathcal{DR}(\mathcal{T})$ is an algebra of the type $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$. Namely we have a natural isomorphism*

$$\mathcal{DR}(\mathcal{T}) \cong \mathcal{O}_{\mathcal{T}}(\mathcal{T},\mathcal{T}).$$

More generally, for every ideals $\mathcal{K}$, $\mathcal{J}$ in $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K})$ the mappings

$$\mathcal{K}(n,m) \ni a \longmapsto i^{\mathcal{M}}_{(n,m)}(a) \in \mathcal{K}_{\mathcal{J}}$$

where $\{i^{\mathcal{M}}_{(n,m)}\}_{n,m\in\mathbb{N}}$ is the representation from Proposition 4.10, induce an isomorphism

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}) \cong \mathcal{DR}(\mathcal{K}_{\mathcal{J}}).$$

Proof. An immediate consequence of Theorem 4.12 and the definition of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ presented below Theorem 4.12. ∎

Corollary 5.2. *A C^*-algebra is a Doplicher-Roberts algebra associated with a C^*-category (as defined in [12]) if and only if it is an algebra of type $\mathcal{O}(\mathcal{T},\mathcal{J})$ for a right tensor C^*-category $\mathcal{T}$.*

Proof. If $\mathcal{T}$ is a C^*-category, then the right tensor C^*-precategory $\mathcal{T}_{\mathcal{J}}$ defined in Theorem 4.12 (where $\mathcal{K} = \mathcal{T}$) is a C^*-category. ∎

It follows that classes of C^*-algebras of type $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ and type $\mathcal{DR}(\mathcal{T})$ coincide (in fact they are both equal to the class of all C^*-algebras). Obviously what distinguishes these algebras is an additional structure - particularly the associated gauge action. We show that an arbitrary circle action may be viewed as a gauge action induced by our construction.
Let $\gamma : S^1 \to \operatorname{Aut} B$ be a circle action on a C^*-algebra B and let $\{B_n\}_{n\in\mathbb{Z}}$ be the family of spectral spaces:

$$B_n = \{b \in B : \gamma_z(b) = z^n b \ \text{ for } \ z \in S^1\}.$$

We set $\mathcal{T} = \{\mathcal{T}(n,m)\}_{n,m\in\mathbb{N}}$ where $\mathcal{T}(r, r+k) = B_k$ and $\mathcal{T}(r+k, r) = B_{-k}$ for all $r, k \in \mathbb{N}$. Then $\mathcal{T}$ with obvious operations is a C^*-precategory, matricially presented by

$$\mathcal{T} = \begin{pmatrix} B_0 & B_1 & B_2 & \cdots \\ B_{-1} & B_0 & B_2 & \cdots \\ B_{-2} & B_{-1} & B_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{33}$$

We may equip $\mathcal{T}$ with a right tensoring $\otimes 1$ which slides elements along the diagonals without changing them. In this way $\mathcal{T}$ becomes a right tensor C^*-precategory such that we have a natural gauge invariant isomorphism

$$B \cong \mathcal{O}_{\mathcal{T}}(\mathcal{T},\mathcal{T}) = \mathcal{DR}(\mathcal{T}).$$

Thus we get

Theorem 5.3. *Every C^*-algebra with a circle action is gauge invariantly isomorphic to an algebra of type $\mathcal{DR}(\mathcal{T})$ (and all the more to an algebra of type $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$).*

The above statement shows that taking into account gauge action the class of relative Cuntz-Pimsner algebras $\mathcal{O}(J,X)$ is strictly smaller than the class of algebras of type $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$. Indeed, we have

Theorem 5.4 (Thm. 3.1 [1])**.** *A C^*-algebra with a circle action γ is gauge invariantly isomorphic to a relative Cuntz-Pimsner algebra $\mathcal{O}(J,X)$ if and only if the action γ is semi-saturated.*

The right tensor C^*-precategory (33) has an advantage that the universal representation embeds the spaces $\mathcal{T}(0,k)$, $k\in\mathbb{N}$, into $\mathcal{O}_{\mathcal{T}}(\mathcal{T},\mathcal{T})$ as spectral subspaces. We now present necessary and sufficient conditions assuring such a property in the general case of algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$. We start with a statement that illustrates the forthcoming definition.

Proposition 5.5. *Let $\mathcal{K}$ be an ideal in a right tensor C^*-precategory $\mathcal{T}$. The following conditions are equivalent.*

i) *There is an action $\mathcal{L}:\{\mathcal{K}(n,m)\}_{n,m=1}^{\infty}\to\mathcal{K}$ where $\mathcal{L}:\mathcal{K}(n+1,m+1)\to\mathcal{K}(n,m)$ is a mapping satisfying the properties*

$$\mathcal{L}(a)\otimes 1 = a \qquad (\textit{right-inversion of } \otimes 1),$$

$$\mathcal{L}(a(b\otimes 1)) = \mathcal{L}(a)b \qquad (\textit{transfer operator property}),$$

for all $a\in\mathcal{K}(n+1,m+1)$, $b\in\mathcal{K}(k,n)$, $n,m,k\in\mathbb{N}$.

ii) *The morphism $\otimes 1: J(\mathcal{K})\cap(\ker\otimes 1)^{\perp}\longrightarrow\{\mathcal{K}(n,m)\}_{n,m=1}^{\infty}$ is an epimorphism (then it is automatically an isomorphism).*

If these conditions are satisfied, then the action $\mathcal{L}$ in item i) is determined uniquely: it is the inverse of the isomorphism $\otimes 1: J(\mathcal{K})\cap(\ker\otimes 1)^{\perp}\longrightarrow\{\mathcal{K}(n,m)\}_{n,m=1}^{\infty}$.

Proof. i) $\Longrightarrow$ ii). Let $a\in\mathcal{K}(n+1,m+1)$. Since $\mathcal{L}(a)\otimes 1=a$ we get $\mathcal{L}(a)\in J(\mathcal{K})(n,m)$. For every $b\in(\mathcal{K}\cap\ker\otimes 1)(k,n)$ we have

$$\mathcal{L}(a)b=\mathcal{L}(a(b\otimes 1))=\mathcal{L}(0)=\mathcal{L}(0(0\otimes 1))=\mathcal{L}(0)\cdot 0=0.$$

Hence $\mathcal{L}(a)\in(\ker\otimes 1)^{\perp}$ and it follows that the image of $\mathcal{L}$ is contained in $J(\mathcal{K})\cap(\ker\otimes 1)^{\perp}$. This together with the equality $\mathcal{L}(a)\otimes 1=a$ implies that $\otimes 1: J(\mathcal{K})\cap(\ker\otimes 1)^{\perp}\longrightarrow\{\mathcal{K}(n,m)\}_{n,m=1}^{\infty}$ is a surjection. Furthermore, since $\otimes 1$ is isometric on $(\ker\otimes 1)^{\perp}$, the image of $\mathcal{L}$ coincides with $J(\mathcal{K})\cap(\ker\otimes 1)^{\perp}$ and $\mathcal{L}$ coincides with the inverse of the isomorphism $\otimes 1: J(\mathcal{K})\cap(\ker\otimes 1)^{\perp}\longrightarrow\{\mathcal{K}(n,m)\}_{n,m=1}^{\infty}$.
ii) $\Longrightarrow$ i). Define $\mathcal{L}$ as the inverse of the isomorphism $\otimes 1: J(\mathcal{K})\cap(\ker\otimes 1)^{\perp}\to\{\mathcal{K}(n,m)\}_{n,m=1}^{\infty}$. Then relation $\mathcal{L}(a)\otimes 1=a$ is trivially satisfied. To show the "transfer operator property" let $a\in\mathcal{K}(n+1,m+1)$ and $b\in\mathcal{K}(k,n)$, $n,m,k\in\mathbb{N}$. Then

$$\mathcal{L}(a(b\otimes 1))\otimes 1=a(b\otimes 1)=(\mathcal{L}(a)\otimes 1)(b\otimes 1)=(\mathcal{L}(a)b)\otimes 1$$

and since both $\mathcal{L}(a(b\otimes 1))$ and $\mathcal{L}(a)b$ belong to $J(\mathcal{K})\cap(\ker\otimes 1)^{\perp}(k,n)$, we get $\mathcal{L}(a(b\otimes 1))=\mathcal{L}(a)b$. ■

Definition 5.6. We shall say that an ideal $\mathcal{K}$ in a right tensor C^*-precategory *admits a transfer action* if it satisfies the equivalent conditions of Proposition 5.5.

The role of the above introduced notion is explained by the following

Theorem 5.7. *Let $\mathcal{K}$ and $\mathcal{J}$ be ideals in a right tensor C^*-precategory $\mathcal{T}$ such that $\mathcal{J}\subset J(\mathcal{K})$. The universal representation embeds each space $\mathcal{K}(0,k)$, $k\in\mathbb{N}$, into $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ as the k-th spectral subspace for the gauge action γ if and only if*

$$\mathcal{J}=J(\mathcal{K})\cap(\ker\otimes 1)^{\perp}$$

and the ideal $\mathcal{K}$ admits a transfer action.

Proof. Assume that $\mathcal{J} = J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}$ and $\mathcal{K}$ admits a transfer action $\mathcal{L}$. By Proposition 4.19 the universal representation $\{i_{(n,m)}\}_{n,m\in\mathbb{N}}$ is faithful. The space $\mathcal{K}(0,k)$ embeds as the k-spectral subspace of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ because of the equality

$$\sum_{j=0}^{r_0} i_{(j,j+k)}(a_{j,j+k}) = i_{(0,k)}\Big(\sum_{j=0}^{r_0} \mathcal{L}^j(a_{j,j+k})\Big), \qquad a_{j,j+k} \in \mathcal{K}(j,j+k),\ r_0 \in \mathbb{N},$$

which follows from Proposition 4.16 and relations, $s = 0, ..., r_0$,

$$\sum_{j=0}^{s} a_{j,j+k} \otimes 1^{s-j} - \Big(\sum_{j=0}^{r_0} \mathcal{L}^j(a_{j,j+k})\Big) \otimes 1^s = \sum_{j=s+1}^{r_0} \mathcal{L}^{j-s}(a_{j,j+k}) \in \mathcal{J}(s,s+k).$$

Now suppose that the representation $\{i_{(n,m)}\}_{n,m\in\mathbb{N}}$ embeds $\mathcal{K}(0,k)$, $k \in \mathbb{N}$, as the k-spectral subspace of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$. Then $\mathcal{J} \subset J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}$ by Corollary 4.17. To show the converse inclusion let $a \in J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}(n,n)$. Since $i_{(n,n)}(a)$ and $i_{(n+1,n+1)}(a \otimes 1)$ lie in the spectral subspace $i_{(0,0)}(\mathcal{K}(0,0))$ there exists $b \in \mathcal{K}(0,0)$ such that $i_{(0,0)}(b) = i_{(n,n)}(a) - i_{(n+1,n+1)}(a \otimes 1)$. Using Proposition 4.16 we get

$$b \otimes 1^n - a \in \mathcal{J}(n,n), \qquad b \otimes 1^{n+1} = 0$$

It follows that $b\otimes 1^n$ belongs both to $(\ker \otimes 1)^{\perp}(n,n)$ and $b\otimes 1^n \in \ker \otimes 1(n,n)$. Hence $b \otimes 1^n = 0$ and consequently $a \in \mathcal{J}(n,n)$. This shows that $\mathcal{J} = J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}$ and to prove that $\otimes 1 : J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp} \longrightarrow \{\mathcal{K}(n,m)\}_{n,m=1}^{\infty}$ is an epimorphism let $a \in \mathcal{K}(n+1,n+r+1)$. Then there exists $b \in \mathcal{K}(0,r)$ such that $i_{(0,0)}(b) = i_{(n+1,n+r+1)}(a)$ or equivalently

$$b \otimes 1^k \in \mathcal{J}(k,k+r), \quad k = 0,...,n, \qquad b \otimes 1^{n+1} = a.$$

In particular, $a = c \otimes 1$ where $c = b \otimes 1^n \in \mathcal{J}(n,n+r) = J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}(n,n+r)$. ■

We now interpret this result on the level of C^*-algebras associated with C^*-correspondences.

Proposition 5.8. *An ideal $\mathcal{K}_X = \{\mathcal{K}(X^{\otimes n}, X^{\otimes m})\}_{n,m\in\mathbb{N}}$ in a right tensor C^*-precategory $\mathcal{T} = \mathcal{T}_X$ associated with a C^*-correspondence X (over a C^*-algebra A) admits a transfer action if and only if X is a Hilbert A-bimodule.*

Proof. Apply Proposition 1.10 iii) and Proposition 5.5 ii). ■

Thus as a corollary to Theorem 5.7 we get

Theorem 5.9 (cf. Prop. 5.17 [17]). *The universal representation of a C^*-correspondence X over A embeds A into a relative Cuntz-Pimsner algebra $\mathcal{O}(J,X)$ as the spectral subspace if and only if X is a Hilbert bimodule over A and*

$$J = (\ker \phi)^{\perp} \cap J(X).$$

If this is the case, then $(\ker \phi)^{\perp} \cap J(X) = \overline{{}_A\langle X, X\rangle}$ and $\mathcal{O}(J,X)$ is canonically isomorphic to the crossed-product $A \rtimes_X \mathbb{Z}$ by Hilbert bimodule X in the sense of [1]. *In particular, each space $X^{\otimes n}$, $n \in \mathbb{N}$, embeds into $\mathcal{O}(J,X)$ as the spectral subspace.*

Proof. If $J = (\ker \phi)^{\perp} \cap J(X)$ and X is a Hilbert bimodule over A, then by Theorem 5.7 algebra A and spaces $X^{\otimes n}$, $n = 1,2,3,...$, embed into $\mathcal{O}(J,X)$ as spectral subspaces. Conversely, if A embeds into $\mathcal{O}(J,X)$ as spectral subspace, then the argument form the proof of Theorem 5.7 shows that $J = (\ker \phi)^{\perp} \cap J(X)$ and X comes from Hilbert bimodule, cf. Proposition 1.10. ■

By Examples 1.11, 3.26 and paragraphs 4.7, 4.8 we get

Corollary 5.10. *Let $\varphi : A \to M(A_0)$ be a partial morphism. Algebra A embeds into the crossed product $A \rtimes_\varphi \mathbb{N}$ as the 0-spectral subspace if and only if φ arises from a partial automorphism (in our broader sense, see Example 1.11). If this is the case, then $A \rtimes_\varphi \mathbb{N}$ coincides with partial crossed-product as defined in* [9] *(where instead of the ideal J one puts hereditary subalgebra A_0).*

Corollary 5.11. *Let $\alpha : A \to A$ be an endomorphism of a unital C^*-algebra A. Algebra A embeds into the relative crossed product $C^*(\mathcal{A}, \alpha; J)$ as the 0-spectral subspace if and only if α admits a complete transfer operator $\mathcal{L}$ (equivalently* $\ker\alpha$ *is unital and $\alpha(A)$ is hereditary subalgebra of A) and $J = \mathcal{L}(A) = (\ker\alpha)^\perp$. If this is the case $C^*(\mathcal{A}, \alpha; J)$ coincides with the crossed product $A \rtimes_\alpha \mathbb{Z}$ introduced in* [3]*, cf.* [23].

By Example 3.27 and paragraph 4.6 we get

Corollary 5.12. *Let $E = (E^0, E^1, r, s)$ be a directed graph and let $R(E) = \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\}$. Then $A = C_0(E^0)$ embeds into the relative graph algebra $C^*(E, V)$ as the spectral subspace if and only if $V = R(E)$ and r, s are injective. In this case $C^*(E, V)$ coincides both with the graph algebra $C^*(E)$ and the partial crossed-product defined by the partial homeomorphism $s \circ r^{-1} : r(E^1) \to s(E^1)$.*

6. Structure Theorem for $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$

In this section we generalize the main result of [10] - Structure Theorem. To this end we introduce and discuss the notions of invariance and saturation for ideals in right tensor C^*-precategories.

Definition 6.1. We shall say that an ideal $\mathcal{N}$ in a right tensor C^*-precategory $\mathcal{T}$ is *invariant* if $\mathcal{N} \otimes 1 \subset \mathcal{N}$. For such an ideal there is a *quotient right tensor C^*-precategory $\mathcal{T}/\mathcal{N}$* defined by

$$(\mathcal{T}/\mathcal{N})(n,m) := \mathcal{T}(n,m)/\mathcal{N}(n,m), \quad \big(a + (\mathcal{T}/\mathcal{N})(n,m)\big) \otimes 1 := a \otimes 1 + (\mathcal{T}/\mathcal{N})(n,m),$$

where $a \in \mathcal{T}(n,m)$, $n, m \in \mathbb{N}$, cf. Proposition 2.10.

Proposition 6.2. *If π is a right tensor representation of an ideal $\mathcal{K}$ in a right tensor C^*-precategory $\mathcal{T}$, then there exists an invariant ideal $\mathcal{N}$ in $\mathcal{T}$ such that*

$$\ker\pi = \mathcal{N} \cap \mathcal{K}.$$

In particular π factors through to the faithful right tensor representation of the ideal $\mathcal{K}/\mathcal{N}$ in the quotient right tensor C^*-precategory $\mathcal{T}/\mathcal{N}$. Moreover, every ideal of the form $\mathcal{N} \cap \mathcal{K}$ where $\mathcal{N}$ is invariant is the kernel of a certain right tensor representation of $\mathcal{K}$.

Proof. Let $\overline{\pi}$ be the extension of π to a right tensor representation of $\mathcal{T}$ defined in Proposition 3.8, and put $\mathcal{N} = \ker\overline{\pi}$. This ideal is invariant because if $\overline{\pi}_{n,m}(a) = 0$, $a \in \mathcal{T}(n,m)$, then by Proposition 3.8 iii) we have $\overline{\pi}_{n+1,m+1}(a) = \overline{\pi}_{n,m}(a)P_{m+1} = 0$. To see the second part of the assertion take any faithful right tensor representation of $\mathcal{K}/\mathcal{N}$ (such a representation exists by Corollary 4.17) and compose it with the quotient homomorphism $q_{\mathcal{N}} : \mathcal{K} \to \mathcal{K}/\mathcal{N}$. ■

We see that invariant ideals are closely related to kernels of right tensor representations. We reveal a corresponding relationship between saturated ideals and ideals of coisometricity for right tensor representations.

Notational conventions 6.3. If $\mathcal{S}$ is a sub-C^*-precategory of $\mathcal{T}$ and $\mathcal{K}$ is an ideal in $\mathcal{T}$, then we shall denote by $\mathcal{S}+\mathcal{K}$ a sub-C^*-precategory of $\mathcal{T}$ where $(\mathcal{S}+\mathcal{K})(n,m) := \mathcal{S}(n,m)+\mathcal{K}(n,m)$, $n,m\in\mathbb{N}$. Similarly, if $\{\mathcal{S}_k\}_{k\in\mathbb{N}}$ is a family of ideals in $\mathcal{T}$, we denote by $\sum_{k=0}^{\infty}\mathcal{S}_k$ the ideal in $\mathcal{T}$ where $(\sum_{k=0}^{\infty}\mathcal{S}_k)(n,m) := \overline{\text{span}}\{a\in\mathcal{S}_k(n,m) : k\in\mathbb{N}\}$. Moreover, if a is a morphism from $\mathcal{T}(n,m)$, then to say that a is in $\mathcal{S}(n,m)$ we shall briefly write $a\in\mathcal{S}$.

Definition 6.4. Let $\mathcal{N}$ and $\mathcal{J}$ be ideals in a right tensor C^*-precategory $\mathcal{T}$. We shall say that $\mathcal{N}$ is *$\mathcal{J}$-saturated* if $\mathcal{J}\cap\otimes 1^{-1}(\mathcal{N})\subset\mathcal{N}$. In general we put

$$\mathcal{S}_{\mathcal{J}}(\mathcal{N}) := \sum_{k=0}^{\infty}\mathcal{S}_k \qquad \text{where } \ \mathcal{S}_0 := \mathcal{N} \ \text{ and } \ \mathcal{S}_k := \mathcal{J}\cap\otimes 1^{-1}(\mathcal{S}_{k-1}),\ k>0.$$

Clearly $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ is the smallest $\mathcal{J}$-saturated ideal containing $\mathcal{N}$ and we shall call it *$\mathcal{J}$-saturation* of $\mathcal{N}$.

The saturation works well with invariance.

Lemma 6.5. *If $\mathcal{N}$ and $\mathcal{J}$ are ideals in a right tensor C^*-precategory $\mathcal{T}$ and $\mathcal{N}$ is invariant, then the $\mathcal{J}$-saturation $S_{\mathcal{J}}(\mathcal{N})$ of $\mathcal{N}$ is invariant and $a\in\mathcal{T}(n,m)$ is in $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ if and only if*

$$(34)\qquad a\otimes 1^k\in\mathcal{J}+\mathcal{N},\quad \text{for all } k\in\mathbb{N},\qquad \text{and}\qquad \lim_{k\to\infty}\|q_{\mathcal{N}}(a\otimes 1^k)\|=0.$$

Proof. Invariance of $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ is straightforward. Passing to the quotient right tensor C^*-precategory $\mathcal{T}/\mathcal{N}$ one easily sees that every element in $\mathcal{T}(n,m)$ satisfying (34) may be approximated (arbitrarily closely) by elements $a\in\mathcal{T}(n,m)$ such that

$$(35)\qquad a\otimes 1^k\in\mathcal{J}+\mathcal{N},\ \text{ for } k=0,..,N-1,\ \text{ and }\ a\otimes 1^N\in\mathcal{N},\ N\in\mathbb{N}.$$

We claim that a satisfies (35) iff $a\in\mathcal{S}_0+\mathcal{S}_1+...+\mathcal{S}_N$ and consequently a satisfies (34) iff $a\in\mathcal{S}_{\mathcal{J}}(\mathcal{N})$. Indeed, let $a\in\mathcal{T}(n,m)$ satisfy (35). Then $a\otimes 1^N\in\mathcal{S}_0=\mathcal{N}$ and

$$a\otimes 1^{N-1}\in\Big(\otimes 1^{-1}(\mathcal{N})\cap(\mathcal{J}+\mathcal{N})\Big)=(\otimes 1^{-1}(\mathcal{N})\cap\mathcal{J})+\mathcal{N}=\mathcal{S}_1+\mathcal{S}_0.$$

Similarly, for $k=1,..,N$ one sees that

$$a\otimes 1^{N-k}\in\mathcal{S}_0+\mathcal{S}_1+...+\mathcal{S}_k \quad\text{ and }\quad a\otimes 1^{N-k-1}\in\mathcal{N}+\mathcal{J}$$

implies that $a\otimes 1^{N-k-1}\in\mathcal{S}_0+\mathcal{S}_1+...+\mathcal{S}_k+\mathcal{S}_{k+1}$. In particular we get $a\in\sum_{k=0}^{N}\mathcal{S}_k$. Conversely, let $a\in(\sum_{k=0}^{N}\mathcal{S}_k)(n,m)$. Then, since $\mathcal{S}_{\mathcal{J}}(\mathcal{N})\subset\mathcal{J}+\mathcal{N}$, we have $a\otimes 1^k\in\mathcal{J}+\mathcal{N}$, $k=0,..,N-1$, and since $\mathcal{S}_0\otimes 1\subset\mathcal{S}_0=\mathcal{N}$ and $\mathcal{S}_k\otimes 1\subset\mathcal{S}_{k-1}$, $k>0$, we get $a\otimes 1^N\in\mathcal{S}_0=\mathcal{N}$. This proves our claim and the lemma. ■

Proposition 6.6. *If $\mathcal{J}\subset J(\mathcal{K})$ is an ideal of coisometricity for a right tensor representation π of an ideal $\mathcal{K}$ in a right tensor C^*-precategory $\mathcal{T}$, then $\ker\pi$ is $\mathcal{J}$-saturated and there exists an invariant $\mathcal{J}$-saturated ideal $\mathcal{N}$ in $\mathcal{T}$ such that*

$$\ker\pi=\mathcal{N}\cap\mathcal{K}.$$

Every ideal of the form $\mathcal{N}\cap\mathcal{K}$ where $\mathcal{N}$ is $\mathcal{J}$-saturated and invariant, is the kernel of a certain right tensor representation of $\mathcal{K}$ whose ideal of coisometricity is $\mathcal{J}\subset J(\mathcal{K})$.

Proof. That $\ker\pi$ is $\mathcal{J}$-saturated follows directly from Definitions 3.18, 6.4. By Proposition 6.2 there exists an invariant ideal $\mathcal{N}$ in $\mathcal{T}$ such that $\ker\pi=\mathcal{N}\cap\mathcal{K}$ and in view of Lemma 6.5 we may assume (if necessary passing to $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$) that $\mathcal{N}$ is also $\mathcal{J}$-saturated. ■

Now we are in a position to prove the main result of this section.

Theorem 6.7 (Structure Theorem)**.** *Let $\mathcal{K}$, $\mathcal{J}$ and $\mathcal{N}$ be ideals in a C^*-precategory $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K})$ and $\mathcal{N}$ is invariant. The subspace*

$$\mathcal{O}(\mathcal{N}) = \overline{\operatorname{span}}\{i_{(m,n)}(a) : a \in (\mathcal{K} \cap \mathcal{N})(n,m),\ n,m \in \mathbb{N}\} \subset \mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$$

generated by the image of $\mathcal{K}\cap\mathcal{N}$ under the universal representation $i = \{i_{(m,n)}\}_{n,m\in\mathbb{N}}$ is an ideal in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ and there are natural isomorphisms

$$\mathcal{O}(\mathcal{N}) \cong \mathcal{O}_{\mathcal{T}}(\mathcal{K}\cap\mathcal{N}, \mathcal{J}\cap\mathcal{N}), \qquad \mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})/\mathcal{O}(\mathcal{N}) \cong \mathcal{O}_{\mathcal{T}/\mathcal{N}}(\mathcal{K}/\mathcal{N}, \mathcal{J}/\mathcal{N}). \tag{36}$$

Moreover, for the $\mathcal{J}$-saturation $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ of $\mathcal{N}$ we have

$$\mathcal{K} \cap \mathcal{S}_{\mathcal{J}}(\mathcal{N}) = i^{-1}(\mathcal{O}(\mathcal{N})).$$

Hence $\mathcal{O}(\mathcal{S}_{\mathcal{J}}(\mathcal{N})) = \mathcal{O}(\mathcal{N})$ and in the right hand sides of (36) the invariant ideal $\mathcal{N}$ may be replaced by the invariant and $\mathcal{J}$-saturated ideal $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$.

Proof. Using operations (23) – (26) we see that

$$\mathcal{M}_{\mathcal{T}}(\mathcal{K}\cap\mathcal{N}) = \operatorname{span}\{i_{(m,n)}(a) : a \in (\mathcal{K}\cap\mathcal{N})(n,m),\ n,m \in \mathbb{N}\}$$

is a two-sided ideal in the algebra $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ defined in Subsection 4.2.1. In particular, it follows that $\mathcal{O}(\mathcal{N}) = \overline{\operatorname{span}}\{i_{(m,n)}(a) : a \in (\mathcal{K}\cap\mathcal{N})(n,m),\ n,m \in \mathbb{N}\}$ is the ideal in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$. To prove that $\mathcal{O}(\mathcal{N}) \cong \mathcal{O}(\mathcal{K}\cap\mathcal{N}, \mathcal{J}\cap\mathcal{N})$ we need to show that the seminorms $\|\cdot\|_{\mathcal{J}}$ and $\|\cdot\|_{\mathcal{J}\cap\mathcal{N}}$ give the same quotients of $\mathcal{M}_{\mathcal{T}}(\mathcal{K}\cap\mathcal{N})$, cf. Proposition 4.13. Clearly, for $a \in \mathcal{M}_{\mathcal{T}}(\mathcal{K}\cap\mathcal{N})$, $\|a\|_{\mathcal{J}\cap\mathcal{N}} = 0$ implies $\|a\|_{\mathcal{J}} = 0$. Conversely, we may assume that $a = \sum_{s=0}^{r} i_{(s,s)}(a_{s,s})$, $a_{s,s} \in (\mathcal{K}\cap\mathcal{N})(s,s)$, $s = 0,...,r$, $r \in \mathbb{N}$, and then the condition $\|a\|_{\mathcal{J}} = 0$ is equivalent to

$$\sum_{j=0}^{s} a_{j,j} \otimes 1^{s-j} \in \mathcal{J}(s,s), \quad s = 1,...,r-1, \qquad \lim_{r\to\infty} \sum_{j=0}^{r} a_{j,j} \otimes 1^{r-j} = 0.$$

By invariance of $\mathcal{N}$ we get

$$\sum_{j=0}^{s} a_{j,j} \otimes 1^{s-j} \in \mathcal{J}\cap\mathcal{N}(s,s), \quad s = 1,...,r-1, \qquad \lim_{r\to\infty} \sum_{j=0}^{r} a_{j,j} \otimes 1^{r-j} = 0$$

which is equivalent to $\|a\|_{\mathcal{J}\cap\mathcal{N}} = 0$. Hence $\mathcal{O}(\mathcal{N}) \cong \mathcal{O}(\mathcal{K}\cap\mathcal{N}, \mathcal{J}\cap\mathcal{N})$.
To construct the isomorphism $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})/\mathcal{O}(\mathcal{N}) \cong \mathcal{O}_{\mathcal{T}/\mathcal{N}}(\mathcal{K}/\mathcal{N}, \mathcal{J}/\mathcal{N})$ consider the right tensor representation $\pi = \{\pi_{nm}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ in $\mathcal{O}_{\mathcal{T}/\mathcal{N}}(\mathcal{K}/\mathcal{N}, \mathcal{J}/\mathcal{N})$ given by $\pi_{nm} = i_{(n,m)} \circ q_{\mathcal{N}}$, $m,n \in \mathbb{N}$, where $i = \{i_{(n,m)}\}_{nm\in\mathbb{N}}$ is the universal representation of $\mathcal{K}/\mathcal{N}$ in $\mathcal{O}_{\mathcal{T}/\mathcal{N}}(\mathcal{K}/\mathcal{N}, \mathcal{J}/\mathcal{N})$. Since for $a \in \mathcal{J}(m,n)$ we have

$$\begin{aligned} q_{\mathcal{N}}(a) \in (\mathcal{J}/\mathcal{N})(m,n) &\iff i_{(n,m)}(q_{\mathcal{N}}(a)) = i_{(n+1,m+1)}(q_{\mathcal{N}}(a) \otimes 1) \\ &\iff \pi_{nm}(a) = \pi_{n+1,m+1}(a \otimes 1), \end{aligned}$$

it follows that π is coisometric on $\mathcal{J}$ and thereby induces a homomorphism Ψ_π of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ onto $\mathcal{O}_{\mathcal{T}/\mathcal{N}}(\mathcal{K}/\mathcal{N}, \mathcal{J}/\mathcal{N})$. Plainly, Ψ_π is zero on $\mathcal{O}(\mathcal{N})$ and hence it factors through to the epimorphism $\Psi : \mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})/\mathcal{O}(\mathcal{N}) \longrightarrow \mathcal{O}_{\mathcal{T}/\mathcal{N}}(\mathcal{K}/\mathcal{N}, \mathcal{J}/\mathcal{N})$. One proves injectivity of Ψ by constructing its inverse. Indeed, the formula

$$\omega_{nm}(a + \mathcal{N}(m,n)) = q(i_{(n,m)}(a)), \qquad a \in (\mathcal{K}/\mathcal{N})(m,n),\ \ n,m \in \mathbb{N},$$

where $q : \mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}) \to \mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})/\mathcal{O}(\mathcal{N})$ is the quotient map, defines a right tensor representation ω of $\mathcal{K}/\mathcal{N}$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})/\mathcal{O}(\mathcal{N})$ which is coisometric on $\mathcal{J}/\mathcal{N}$. Thus ω integrates to a homomorphism from $\mathcal{O}_{\mathcal{T}/\mathcal{N}}(\mathcal{K}/\mathcal{N}, \mathcal{J}/\mathcal{N})$ to $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})/\mathcal{O}(\mathcal{N})$ which is inverse to Ψ.

Furthermore, for the representation π defined above, we have $\ker \pi_{nm} = i^{-1}_{(n,m)}(\mathcal{O}(\mathcal{N}))$ and thus by Corollary 4.17, $a \in \mathcal{K}(n,m)$ is in $i^{-1}_{(n,m)}(\mathcal{O}(\mathcal{N}))$ if and only if

$$a \otimes 1^k \in \mathcal{J} + \mathcal{N}, \quad \text{for all} \quad k \in \mathbb{N}, \qquad \text{and} \qquad \lim_{k\to\infty} \|q_{\mathcal{N}}(a \otimes 1^k)\| = 0.$$

In view of Lemma 6.5 this proves the second part of the theorem. ■

The Structure Theorem has a number of important consequences. Firstly, $\mathcal{O}(\mathcal{N})$ is a *gauge invariant ideal* in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$, i.e. it is globally invariant under the associated gauge action, and if $\mathcal{N}$ is $\mathcal{J}$-saturated the ideal $\mathcal{K}\cap\mathcal{N}$ is uniquely determined by $\mathcal{O}(\mathcal{N})$. Hence, denoting by $\mathrm{Lat}_{\mathcal{J}}(\mathcal{K})$ the lattice of ideals of the form $\mathcal{K}\cap\mathcal{N}$ where $\mathcal{N}$ is an invariant and $\mathcal{J}$-saturated, and by $\mathrm{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}))$ the lattice of gauge invariant ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ we get the natural embedding

$$\mathrm{Lat}_{\mathcal{J}}(\mathcal{K}) \subset \mathrm{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})).$$

In general this embedding is not an isomorphism. However, we will show in Theorem 7.8 that in certain important cases we have $\mathrm{Lat}_{\mathcal{J}}(\mathcal{K}) \cong \mathrm{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}))$.

Secondly, the zero ideal $\mathcal{N}=\{0\}$ is invariant and hence dividing $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ by the ideal generated by the $\mathcal{J}$-saturation $\mathcal{S}_{\mathcal{J}}(\{0\})$ of $\{0\}$ we reduce the relations defining $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ without affecting the algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ itself. We formalize this remark as follows.

Definition 6.8. Let $\mathcal{J}$ be an ideal in a right tensor C^*-precategory $\mathcal{T}$. We denote the $\mathcal{J}$-saturation $\mathcal{S}_{\mathcal{J}}(\{0\})$ of the zero ideal by $\mathcal{R}_{\mathcal{J}}$ and call it a *reduction ideal* associated with $\mathcal{J}$.

Theorem 6.9 (Reduction of relations)**.** *Let $\mathcal{K}$ be an ideal in a right tensor C^*-category $\mathcal{T}$ and let $\mathcal{J}$ be an ideal in $\mathcal{J}(\mathcal{K})$. Putting*

$$\mathcal{T}_{\mathcal{R}} := \mathcal{T}/\mathcal{R}_{\mathcal{J}}, \qquad \mathcal{K}_{\mathcal{R}} := \mathcal{K}/\mathcal{R}_{\mathcal{J}}, \qquad \mathcal{J}_{\mathcal{R}} := \mathcal{J}/\mathcal{R}_{\mathcal{J}},$$

we get a "reduced" right tensor C^*-category $\mathcal{T}_{\mathcal{R}}$, with right tensoring $\otimes 1_{\mathcal{R}}$. The "reduced" ideals $\mathcal{K}_{\mathcal{R}}$, $\mathcal{J}_{\mathcal{R}}$ are such that $\mathcal{J}_{\mathcal{R}} \subset \mathcal{J}(\mathcal{K}_{\mathcal{R}}) \cap (\ker \otimes 1_{\mathcal{R}})^{\perp}$ and there is a natural isomorphism

$$\mathcal{O}(\mathcal{K},\mathcal{J}) \cong \mathcal{O}(\mathcal{K}_{\mathcal{R}},\mathcal{J}_{\mathcal{R}}).$$

Proof. Clear by Theorem 6.7. ■

Remark 6.10. We may often reduce even more relations in $\mathcal{T}$ in the sense that for any invariant ideal $\mathcal{R}'$ in $\mathcal{T}$ such that $\mathcal{R}' = \mathcal{R}_{\mathcal{J}} \cap \mathcal{K}$ we have

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}) \cong \mathcal{O}_{\mathcal{T}_{\mathcal{R}'}}(\mathcal{K}_{\mathcal{R}'},\mathcal{J}_{\mathcal{R}'})$$

where $\mathcal{K}_{\mathcal{R}'} := \mathcal{K}/\mathcal{R}'$ and $\mathcal{J}_{\mathcal{R}'} := \mathcal{J}/\mathcal{R}'$ coincide with $\mathcal{K}_{\mathcal{R}}$ and $\mathcal{J}_{\mathcal{R}}$, respectively, but in general $\mathcal{T}_{\mathcal{R}'} = \mathcal{T}/\mathcal{R}'$ is "smaller" than $\mathcal{T}_{\mathcal{R}} = \mathcal{T}/\mathcal{R}_{\mathcal{J}}$.

6.1. **Structure theorem for relative Cuntz-Pimsner algebras.** As an application of Theorem 6.7 we get a version of Structure Theorem for relative Cuntz-Pimsner algebras which improves [10, Thm 3.1]. For that purpose we fix a C^*-correspondence X over a C^*-algebra A, and establish the relationships between the relevant ideals in $\mathcal{K}_X$ and A.

Proposition 6.11. *Each of the relations*

$$I = (\mathcal{K}_X \cap \mathcal{N})(0,0), \qquad \mathcal{K}_X \cap \mathcal{N} = \mathcal{K}_X(I) \tag{37}$$

establish a one-to-one correspondence between X-invariant ideals I in A and ideals of the form $\mathcal{K}_X \cap \mathcal{N}$ where $\mathcal{N}$ is an invariant ideal in $\mathcal{T}_X$. Moreover we have a natural isomorphism of C^*-precategories

$$\mathcal{K}_X/\mathcal{N} \cong \mathcal{K}_{X/XI} = \{\mathcal{K}\left((X/XI)^{\otimes n}, (X/XI)^{\otimes m}\right)\}_{n,m\in\mathbb{N}}.$$

Proof. One easily sees that if I is X-invariant ideal in A, then $\mathcal{N} := \mathcal{T}_X(I)$ is invariant, and $I = (\mathcal{K}_X \cap \mathcal{N})(0,0)$. Conversely, if $\mathcal{N}$ is an ivariant ideal in $\mathcal{T}_X$, then $I := (\mathcal{K}_X \cap \mathcal{N})(0,0)$ is X-invariant and by Proposition 2.16, $\mathcal{K}_X \cap \mathcal{N} = \mathcal{K}_X(I)$. The remaining part of proposition follows from Lemma 1.7 and Corollary 1.5. ■

The notion we are about to introduce, in the case when $J = J(X) \cap (\ker\phi)^{\perp}$, coincides with the property called X-saturation in [29, Def. 6.1] and negative invariance in [18, Def. 4.8], cf. also [18, Def. 4.14].

Definition 6.12. Let I and J be ideals in A. We shall say that I is *J-saturated* if $J \cap \phi^{-1}(\mathcal{L}(XI)) \subset I$, equivalently if

$$a \in J \quad \text{and} \quad \varphi(a)X \subset XI \implies a \in I.$$

In general we put $S_J(I) := \sum_{k=0}^{\infty} S_k$ where $S_0 := I$ and $S_k := J \cap \phi^{-1}(S_{k-1})$, $k > 0$. Then $S_J(I)$ is the smallest J-saturated ideal containing I which we shall call *J-saturation* of I.

Proposition 6.13. *Let J and I be ideals in A. The $\mathcal{K}_X(J)$-saturation of $\mathcal{K}_X(I)$ coincides with $\mathcal{K}_X(S_J(I))$. In particular, I is J-saturated if and only if $\mathcal{K}_X(I)$ is $\mathcal{K}_X(J)$-saturated.*

Proof. It suffices to check that under notation of Definitions 6.4, 6.12 we have $\mathcal{K}_X(S_n) = \mathcal{S}_n$, $n \in \mathbb{N}$, which is straightforward. ■

In view of the above statement, Corollary 3.22 and Proposition 6.6 we get

Proposition 6.14. *Let $J \subset J(X)$. If J is an ideal of coisometricity for a representation (π, t) of X, then $\ker\pi$ is J-saturated and X-invariant. Conversely, every J-saturated and X-invariant ideal in A is the kernel $\ker\pi$ for a certain representation (π, t) of X whose ideal of coisometricity is J.*

To state the Structure Theorem for C^*-correspondences in its full force we introduce algebras that generalize relative Cuntz-Pimsner algebras.

Definition 6.15. If I and J are ideals in A such that $J \subset J(XI)$ we put

$$\mathcal{O}_X(I,J) := \mathcal{O}_{\mathcal{T}_X}(\mathcal{K}_X(I), \mathcal{K}_X(J)).$$

This algebra is well defined because $\mathcal{K}_X(J) \subset J(\mathcal{K}_X(I))$ iff $J \subset J(XI)$. In particular, we have $\mathcal{O}(J,X) = \mathcal{O}_X(A,J)$.

In Remark 2.18 we noted that ideal structures of $\mathcal{K}_{XI} = \{\mathcal{K}(XI^{\otimes n}, XI^{\otimes m})\}_{n,m\in\mathbb{N}}$ and $\mathcal{K}_X(I) = \{\mathcal{K}(X^{\otimes n}, X^{\otimes m}I)\}_{n,m\in\mathbb{N}}$ are isomorphic. In view of the following lemma, this could be interpreted as that these C^*-precategories are "Morita equivalent" where the "equivalence" is established by $L = \{\mathcal{K}(XI^{\otimes n}, X^{\otimes m})\}_{n,m\in\mathbb{N}}$.

Lemma 6.16. *Let I be an arbitrary ideal in A. Then*

$$L = \{\mathcal{K}(XI^{\otimes n}, X^{\otimes m})\}_{n,m\in\mathbb{N}}$$

is a left ideal in a C^*-precategory $\mathcal{T}_X$, and the following relations hold

$$L^*L = \mathcal{K}_{XI}, \qquad LL^* = \mathcal{K}_X(I),$$

where $L^*L(n,m) = \overline{\operatorname{span}}\{x^*y : x \in L(m,k), y \in L(n,k)\}$ and analogously $LL^*(n,m) = \overline{\operatorname{span}}\{xy^* : x \in L(k,m), y \in L(k,n)\}$, $n,m \in \mathbb{N}$.

Proof. Clearly, $\mathcal{K}_{XI} \subset L$ and hence $\mathcal{K}_{XI} = \mathcal{K}_{XI}^* \mathcal{K}_{XI} \subset L^* L$. The opposite inclusion follows from the fact that for $x, u \in X^{\otimes k}$, $y \in XI^{\otimes n}$ and $v \in XI^{\otimes m}$

$$(\Theta_{u,v})^* \Theta_{x,y} = \Theta_{v, y\langle x,u\rangle} \in \mathcal{K}(XI^{\otimes n}, XI^{\otimes m}).$$

Thus $L^* L = \mathcal{K}_{XI}$. To show that $LL^* = \mathcal{K}_X(I)$ it suffices to apply Proposition 2.16, since both LL^* and $\mathcal{K}_X(I)$ are ideals in $\mathcal{K}_X$ such that $LL^*(0,0) = \mathcal{K}_X(I)(0,0) = I$. ■

When passing to C^*-algebras we have introduced, the above "Morita equivalence" of C^*-precategories yields Morita equivalence of C^*-algebras.

Theorem 6.17 (Morita equivalence of $\mathcal{O}_X(I, I\cap J)$ and $\mathcal{O}_{XI}(I, J\cap I)$)**.** *Let I be an X-invariant ideal in A and let J be an ideal in $J(X)$. The algebras*

$$\mathcal{O}_X(I, J\cap I), \qquad \mathcal{O}_{XI}(I, J\cap I) = \mathcal{O}(J\cap I, XI)$$

may be naturally considered as subalgebras of $\mathcal{O}_{\mathcal{T}_X}(\mathcal{T}_X, \mathcal{K}_X(J\cap I))$. These subalgebras coincide whenever $\varphi(I)X = XI$, and in general they are Morita equivalent with an equivalence established via the subspace

$$\mathcal{L} = \overline{\operatorname{span}}\{i_{(m,n)}(a) : a \in L(n,m)\} \subset \mathcal{O}_{\mathcal{T}_X}(\mathcal{T}_X, \mathcal{K}_X(J\cap I)).$$

Proof. The first part of the statement is straightforward, cf. Definition 6.15. To see the second part consider the algebra $\mathcal{M}_\mathcal{T}(\mathcal{T})$ for $\mathcal{T} = \mathcal{T}_X$ (defined in Subsection 4.2.1). For any sub-C^*-precategory $\mathcal{S}$ of $\mathcal{T}$ we put

$$\mathcal{M}_\mathcal{T}(\mathcal{S}) = \operatorname{span}\{i_{(m,n)}(a) : a \in \mathcal{S}(n,m)\}.$$

Then $\mathcal{M}_\mathcal{T}(\mathcal{K}(I))$ is a two-sided ideal in $\mathcal{M}_\mathcal{T}(\mathcal{T})$, and $\mathcal{M}_\mathcal{T}(L)$ is a left ideal in $\mathcal{M}_\mathcal{T}(\mathcal{T})$ such that

$$\mathcal{M}_\mathcal{T}(L) \star \mathcal{M}_\mathcal{T}(L)^* = \mathcal{M}_\mathcal{T}(\mathcal{K}(I)).$$

Indeed, since $L \subset \mathcal{K}(I)$ we have $\mathcal{M}_\mathcal{T}(L) \star \mathcal{M}_\mathcal{T}(L)^* \subset \mathcal{M}_\mathcal{T}(\mathcal{K}(I))$, and the opposite inclusion follows from Lemma 6.16. Similarly, $\mathcal{M}_\mathcal{T}(\mathcal{K}(XI))$ is a *-subalgebra of $\mathcal{M}_\mathcal{T}(\mathcal{T})$ such that

$$\mathcal{M}_\mathcal{T}(L)^* \star \mathcal{M}_\mathcal{T}(L) = \mathcal{M}_\mathcal{T}(\mathcal{K}(XI)).$$

Indeed, since $\mathcal{K}(XI) \subset L$ we have $\mathcal{M}_\mathcal{T}(\mathcal{K}(I)) \subset \mathcal{M}_\mathcal{T}(L)^* \star \mathcal{M}_\mathcal{T}(L)$. To see the opposite inclusion notice that for any $y \in XI^{\otimes n}$, $v \in XI^{\otimes m}$, $u, x_1 \in X^{\otimes l}$, $x_2 \in X^{\otimes k}$, by X-invariance of I, the operator

$$\left((\Theta_{u,v})^* \otimes 1^k\right) \Theta_{x_1\otimes x_2,\, y} = \Theta_{v\otimes\varphi(\langle u,x_1\rangle)x_2,\, y}$$

is in $\mathcal{K}(XI^{\otimes n}, XI^{\otimes(m+k)})$.
Thus it is enough to take the quotients with respect to seminorm $\|\cdot\|_{\mathcal{K}_X(J\cap I)}$ defined in Proposition 4.13 and then apply the enveloping procedure. ■

We are in a position to formulate the main result of this subsection.

Theorem 6.18 (Structure Theorem for C^*-correspondences)**.** *Suppose J is an ideal in $J(X)$ and let $\mathcal{O}(I)$ denote the ideal in $\mathcal{O}(J,X)$ generated by the image of an X-invariant ideal I under the universal representation. There are natural isomorphisms*

$$\mathcal{O}(I) \cong \mathcal{O}_X(I, J\cap I), \qquad \mathcal{O}(J,X)/\mathcal{O}(I) \cong \mathcal{O}(J/I, X/XI). \tag{38}$$

In particular, algebras $\mathcal{O}(I)$ and $\mathcal{O}(J\cap I, XI)$ are Morita equivalent and if $\phi(I)X = XI$, then simply $\mathcal{O}(I) \cong \mathcal{O}_X(I, J\cap I) = \mathcal{O}(J\cap I, XI)$. Moreover, the J-saturation $S_J(I)$ of I is X-invariant and

$$S_J(I) = i_{00}^{-1}(\mathcal{O}(I)).$$

Thus $\mathcal{O}(I)=\mathcal{O}(S_J(I))$ and in the right hand sides of (38) *the X-invariant ideal I may be replaced by the X-invariant and J-saturated ideal $S_J(I)$.*

Proof. One sees that $\mathcal{O}(I)=\overline{\text{span}}\{i_{(m,n)}(a): a\in\mathcal{K}_X(I)(n,m), n,m\in\mathbb{N}\}$ and hence hypotheses follows from Theorems 6.7, 6.17 and Propositions 6.11, 6.13. ■

Remark 6.19. For an X-invariant ideal I in A the subspace $\phi(I)X$ of X may be considered as a C^*-correspondence over I. The argument from [17, Prop. 9.3] shows that $\mathcal{O}(J\cap I,\phi(I)X)$ is Morita equivalent to $\mathcal{O}(I)$. Thus we have three C^*-algebras with natural embeddings

$$\mathcal{O}(J\cap I,\phi(I)X)\subset\mathcal{O}(J\cap I,XI)\subset\mathcal{O}_X(I,J\cap I)\cong\mathcal{O}(I)$$

which are all Morita equivalent, and if $\phi(I)X=XI$ then they are actually equal. One of advantages of our approach is that we have been able to identify $\mathcal{O}(I)$ precisely as $\mathcal{O}_X(I,J\cap I)=\mathcal{O}_{\mathcal{T}_X}(\mathcal{K}_X(I),\mathcal{K}_X(J\cap I))$ (not only up to Morita equivalence, as it was in [10],[17]).

The reduction procedure from Theorem 6.9 is in full consistency with reduction of relations in C^*-correspondences presented in [25].

Definition 6.20. Let J be an ideal in A. The J-saturation $S_J(\{0\})$ of the zero ideal will be denoted by R_J and called a *reduction ideal* associated to J.

Theorem 6.21 (Reduction of C^*-correspondences)**.** *Let X be a C^*-correspondence over A and J an ideal in $J(X)$. The reduction ideal R_J is X-invariant, and putting*

$$X_R:=X/XR_J,\qquad A_R:=A/R_J,\qquad J_R:=J/R_J,$$

we get a "reduced" C^*-correspondence X_R over A_R such that $J_R\subset J(X_R)\cap\ker(\phi_R)^\perp$ where ϕ_R is the left action on X_R, and

$$\mathcal{O}(J,X)\cong\mathcal{O}(J_R,X_R).$$

Proof. Clear by Theorem 6.18. ■

Example 6.22. Let $X=X_\varphi$ be the C^*-correspondence associated with a partial morphism $\varphi:A\to M(A_0)$. One readily checks that an ideal I in A is X-invariant if and only if $\varphi(I)A_0\subset I$. For an X-invariant ideal I we have a restricted partial morphism $\varphi_I: I\to M(A_0\cap I)$ and a quotient partial morphism $\varphi^I: A/I\to M(A_0/I)$ where

$$\varphi_I=\varphi|_I,\qquad \varphi^I(a+I)(a_0+I):=\varphi(a)a_0+I,\qquad a\in A,\ a_0\in A_0.$$

Both φ_I and φ^I are well defined as $A_0\cap I$ is a hereditary subalgebra of I, A_0/I is a hereditary subalgebra of A/I, $\varphi_I(I)A_0\cap I=A_0\cap I$ and $\varphi^I(A/I)(A_0/I)=A_0/I$. Furthermore, we may naturally identify (as C^*-correspondences) X_{φ^I} with X/XI and X_{φ_I} with $\phi(I)X$. Using our general definition of relative crossed products (Definition 4.9), for an ideal J in $\varphi^{-1}(A_0)=J(X)$, by Theorem 6.18 and Remark 6.19 we get

$$C^*(\varphi;J)/\mathcal{O}(I)\cong C^*(\varphi^I;J/I),$$

where $\mathcal{O}(I)$ is Morita equivalent to $C^*(\varphi_I;J\cap I)$. If we denote by J_∞ the ideal in J consisting of those elements $a\in J$ for which the iterates $\varphi^n(a)$, $n\in\mathbb{N}$, make sense and belong to J, i. e.

$$a\in J_\infty\iff a\in J,\ \ \varphi(a)\in J\cap A_0,\ \ \varphi^2(a)\in J\cap A_0,\ \ ...,\ \ \varphi^n(a)\in J\cap A_0,\ \ ...\ ,$$

then the reduction ideal $R := R_J$ assumes the following form

$$R = \overline{\{a \in J_\infty : \exists_{n\in\mathbb{N}}\ \varphi^n(a) = 0\}},$$

cf. [25]. For the quotient partial morphism $\varphi^R : A/R \to M(A_0/R)$ we have

$$C^*(\varphi, J) \cong C^*(\varphi^R, J/R).$$

Hence φ^R may be viewed as a natural reduction of the partial morphism φ relative to the ideal J.

Example 6.23. Suppose that $X = X_E$ is the C^*-correspondence of a graph E, I is an ideal in $A = C_0(E^0)$ and $F \subset E^0$ is a complement of the hull of I. Then

$$XI = \overline{\text{span}}\{\delta_e : r(e) \in F\}, \qquad \phi(I)X = \overline{\text{span}}\{\delta_e : s(e) \in F\}.$$

It follows that I is X-invariant if and only if V is hereditary, that is if $s(e) \in F \implies r(e) \in F$, for all $e \in E^1$, cf. [7], [6], [29]. When F is hereditary, then (slightly abusing notation) we may consider $E \setminus F := (E^0 \setminus F, r^{-1}(E^0 \setminus F))$ and $F := (F, s^{-1}(F))$ as subgraphs of E. In this event X/XI is canonically isomorphic to $X_{E\setminus F}$ and $\phi(I)X$ is canonically isomorphic to X_F, cf. [10, Ex. 2.4]. We recall that $J(X) = \overline{\text{span}}\{\delta_v : |s^{-1}(v)| < +\infty\}$. Hence Theorem 6.18 imply that for any hereditary subset $F \subset E^0$ and any $V \subset \{v \in E^0 : s^{-1}(v) < \infty\}$ there is an ideal $\mathcal{O}(I)$ in the relative graph algebra $\mathcal{O}(E, V)$ such that

$$\mathcal{O}(E, V)/\mathcal{O}(I) \cong \mathcal{O}(E \setminus F, V \setminus F)$$

and $\mathcal{O}(I)$ is Morita equivalent to the relative graph algebra $C^*(F, F \cap V)$. We shall say that a subset $F \subset E^0$ is *V-saturated* if every vertex in $V \subset E^0$ which feeds into F and only F is in F:

$$v \in V \ \text{ and } \ \{r(e) : s(e) = v\} \subset F \implies v \in F,$$

and by a *V-saturation* of a set F we mean the smallest saturated subset $S_V(F)$ of E^0 containing F. In the case $V = \{v \in E^0 : 0 < s^{-1}(v) < \infty\}$ these notions coincide with the ones (without prefix V) defined in [6], [7]. If $J = \overline{\text{span}}\{\delta_v : v \in V\}$, then the reduction ideal $R := R_J$ is spanned by the point masses of the V-saturation $S_V(\emptyset)$ of the empty set ($S_V(\emptyset)$ consists of vertices in V that form paths leading to sinks). As a consequence we have

$$\mathcal{O}(E, V) \cong \mathcal{O}\left(E \setminus S_V(\emptyset), V \setminus S_V(\emptyset)\right).$$

Hence $E \setminus S_V(\emptyset)$ may be considered as a natural reduction of the graph E relative to the set V.

7. Ideal structure of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$

In this section we prove gauge invariance theorem for $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ and describe the lattice of gauge-invariant ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. These results generalize the corresponding statements for relative Cuntz-Pimsner obtained in [17], [10], [18] (see the relevant discussions in Subsection 7.1 and Section 9).

Let us fix ideals $\mathcal{K}$, $\mathcal{J}$ in a C^*-precategory $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K})$ and let $\mathcal{K}_{\mathcal{J}}$ be the C^*-precategory defined in Theorem 4.12. The representation $\{i^{\mathcal{M}}_{(n,m)}\}_{n,m\in\mathbb{N}}$ from Proposition 4.10 factors through to the injective homomorphism of $\mathcal{K}$ into $\mathcal{K}_{\mathcal{J}}$:

$$\mathcal{K}(m, n) \ni a \longmapsto i^{\mathcal{M}}_{(n,m)}(a) \in \mathcal{K}_{\mathcal{J}}(m, n). \tag{39}$$

We use it to adopt the identifications

$$\mathcal{K} \subset \mathcal{K}_{\mathcal{J}}, \qquad \mathcal{DR}(\mathcal{K}_{\mathcal{J}}) = \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}).$$

cf. Proposition 5.1.

Proposition 7.1. *We have a one-to-one correspondence between the right tensor representations* $\pi = \{\pi_{n,m}\}_{n,m\in\mathbb{N}}$ *of* $\mathcal{K}$ *coisometric on* $\mathcal{J}$ *and right tensor representations* $\widetilde{\pi} = \{\widetilde{\pi}_{n,m}\}_{n,m\in\mathbb{N}}$ *of* $\mathcal{K}_{\mathcal{J}}$ *coisometric on* $\mathcal{K}_{\mathcal{J}}$*. This correspondence is given by*

$$\widetilde{\pi}_{r,r+k}\Big(\sum_{\substack{j=0,\\ j+k\geq 0}}^{r} i^{\mathcal{M}}_{(j,j+k)}(a_{j,j+k})\Big) = \sum_{\substack{j=0,\\ j+k\geq 0}}^{r} \pi_{j,j+k}(a_{j,j+k}). \tag{40}$$

Proof. In view of the definition of $\mathcal{K}_{\mathcal{J}}$ the assertion may be verified directly. One may also deduce it from Theorems 4.11, 4.14 and Proposition 5.1. ∎

It is well known that if two C^*-algebras A, B admit circle actions, then a *-homomorphism $h : A \to B$ that maps faithfully spectral subspaces of A onto the corresponding spectral subspaces of B is an isomorphism if and only if it is gauge invariant. Thus the following statement can be thought of as a (stronger) version of what is usually meant by a gauge invariance theorem.

Definition 7.2. A representation $\pi = \{\pi_{n,m}\}_{n,m\in\mathbb{N}}$ of an ideal $\mathcal{K}$ in a right tensor C^*-precategory is said to admit a gauge action if for every $z \in S^1$ relations

$$\beta_z(\pi_{nm}(a)) = z^{n-m}\pi_{nm}(a), \qquad a \in \mathcal{K}(m,n),\ n,m \in \mathbb{N},$$

give rise to a well defined $*$-homomorphism $\beta_z : C^*(\pi) \to C^*(\pi)$ where $C^*(\pi)$ stands for the C^*-algebra generated by the spaces $\pi_{nm}(\mathcal{K}(m,n))$, $n,m \in \mathbb{N}$.

Theorem 7.3 (Gauge-invariant uniqueness for $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$)**.** *Let* $\pi = \{\pi_{n,m}\}_{n,m\in\mathbb{N}}$ *be a right tensor representation of* $\mathcal{K}$ *coisometric on* $\mathcal{J} \subset J(\mathcal{K})$ *and let* $\mathcal{R}_{\mathcal{J}}$ *be the reduction ideal in* $\mathcal{K}$ *associated to* $\mathcal{J}$ *(Definition 6.8). The following conditions are equivalent*

i) π *integrates to a representation that is faithful on* 0*-spectral subspace of* $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$
ii) π *integrates to a representation that is faithful on all spectral subspaces of* $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$
iii) $\ker\widetilde{\pi} = \mathcal{R}_{\mathcal{J}}$
iv) $\ker\pi = \mathcal{R}_{\mathcal{J}}$ *and*

$$\mathcal{J}(n,n) = \pi_{n,n}^{-1}\left(\sum_{j=1}^{k}\pi_{n+j,n+j}(\mathcal{K}(n+j,n+j))\right), \qquad n,k \in \mathbb{N}. \tag{41}$$

In particular, π *integrates to the faithful representation of* $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ *if and only if* π *admits a gauge action, and one of the equivalent conditions i)-iv) holds.*

Proof. Equivalence i) $\Longleftrightarrow$ ii) follows from C^*-equality and algebraic relations between spectral subspaces, cf. [9]. The representations Ψ_π and $\Psi_{\widetilde{\pi}}$ of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$, arising from π and $\widetilde{\pi}$, coincide and hence we put $\Psi = \Psi_\pi = \Psi_{\widetilde{\pi}}$. From the direct limit construction of $\mathcal{D}R(\mathcal{K}_{\mathcal{J}})$ we see that Ψ is faithful on spectral subspaces iff it is faithful on spaces $\phi_{nm}(\mathcal{K}_{\mathcal{J}}(n,m))$, $n,m \in \mathbb{N}$, where $\{\phi_{nm}\}_{n,m\in\mathbb{N}}$ is a universal representation of $\mathcal{K}_{\mathcal{J}}$ in $\mathcal{D}R(\mathcal{K}_{\mathcal{J}})$. The latter requirement is in turn equivalent to the equality $\ker\widetilde{\pi} = \mathcal{R}_{\mathcal{J}}$. Hence we get ii) $\Longleftrightarrow$ iii). To see that iii) $\Longleftrightarrow$ iv) notice that, in view of (40), equality $\ker\widetilde{\pi} = \mathcal{R}_{\mathcal{J}}$ is equivalent to the equality $\ker\pi = \mathcal{R}_{\mathcal{J}}$ and condition (41). ∎

Corollary 7.4 (Gauge invariance theorem for $\mathcal{DR}(\mathcal{T})$)**.** *A right tensor representation $\pi = \{\pi_{n,m}\}_{n,m\in\mathbb{N}}$ of $\mathcal{T}$ coisometric on $\mathcal{T}$ integrates to faithful representation of $\mathcal{DR}(\mathcal{T})$ if and only if π admits a gauge action and* $\ker = \mathcal{R}_{\mathcal{T}}$.

Remark 7.5. The condition (41) implies that $\mathcal{J}$ is an ideal of coisometricity for π. In many natural situations the ideal of coisometricity automatically satisfies (41), cf. Corollary 7.9 below. On the level of Cuntz-Pimsner algebras condition (41) reduces to equality (53), the role of which was discovered by Katsura [17] and led him to the notion of a T-pair (see Proposition 7.14 and Definition 7.15 below).

Continuing the discussion undertaken below Theorem 6.7, we give a complete description of gauge invariant ideal structure of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$.

Theorem 7.6 (Lattice structure of gauge invariant ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$)**.** *We have a one-to-one correspondences between the following objects*

i) *kernels of right tensor representations $\widetilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ coisometric on $\mathcal{K}_{\mathcal{J}}$*

ii) *invariant and $\mathcal{K}_{\mathcal{J}}$-saturated ideals $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$, that is ideals satisfying*

$$\mathcal{N}_{\mathcal{J}} \otimes_{\mathcal{J}} 1 \subset \mathcal{N}_{\mathcal{J}} \quad \textit{and} \quad (\otimes_{\mathcal{J}} 1)^{-1}(\mathcal{N}_{\mathcal{J}}) \subset \mathcal{N}_{\mathcal{J}}$$

iii) *gauge invariant ideals P in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}) = \mathcal{DR}(\mathcal{K}_{\mathcal{J}})$*

These correspondences preserve inclusions and are given by

$$\mathcal{N}_{\mathcal{J}} = \ker \widetilde{\pi}, \qquad \ker \Psi_{\widetilde{\pi}} = \mathcal{O}(\mathcal{N}_{\mathcal{J}}) = P, \qquad \mathcal{N}_{\mathcal{J}}(n,m) = i_{(n,m)}^{-1}(P), \quad n,m \in \mathbb{N},$$

where $i = \{i_{(n,m)}\}_{n,m\in\mathbb{N}}$ denotes the universal right tensor representation of $\mathcal{K}_{\mathcal{J}}$ in $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})$ and $\mathcal{O}(\mathcal{N}_{\mathcal{J}})$ is the closed linear span of the image of $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})$. In particular,

$$P \cong \mathcal{DR}(\mathcal{N}_{\mathcal{J}}), \qquad P = \overline{\operatorname{span}}\{i_{(m,n)}(a) : a \in \mathcal{N}_{\mathcal{J}}(n,m)\} \subset \mathcal{DR}(\mathcal{K}_{\mathcal{J}}),$$

and we have the lattice isomorphism $\operatorname{Lat}_{\mathcal{K}_{\mathcal{J}}}(\mathcal{K}_{\mathcal{J}}) \cong \operatorname{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}))$.

Proof. If $\widetilde{\pi}$ is a right tensor representation of $\mathcal{K}_{\mathcal{J}}$, then by Proposition 6.6 $\ker\widetilde{\pi}$ is an invariant and $\mathcal{K}_{\mathcal{J}}$-saturated ideal in $\mathcal{K}_{\mathcal{J}}$. Conversely, if $\mathcal{N}_{\mathcal{J}}$ is an invariant and $\mathcal{K}_{\mathcal{J}}$-saturated ideal in $\mathcal{K}_{\mathcal{J}}$, then by Theorem 6.7 we have a gauge invariant homomorphism $\Psi : \mathcal{DR}(\mathcal{K}_{\mathcal{J}}) \to \mathcal{DR}(\mathcal{K}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}})$ whose kernel is $\mathcal{O}(\mathcal{N}_{\mathcal{J}})$. Hence disintegrating Ψ we get a right tensor representation $\widetilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ coisometric on $\mathcal{K}_{\mathcal{J}}$ such that $\mathcal{N}_{\mathcal{J}} = \ker\widetilde{\pi}$. This proves the correspondence between the objects in i) and ii). Let now P be an arbitrary gauge invariant ideal in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}) = \mathcal{DR}(\mathcal{K}_{\mathcal{J}})$. It is clear that spaces $\mathcal{N}_{\mathcal{J}}(m,n) := i_{(n,m)}^{-1}(P)$, $n,m\in\mathbb{N}$, form an ideal $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$. By definition of the algebra $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})$ we have $i_{(n,m)}(a) = i_{(n+1,m+1)}(a\otimes 1)$, for all $a \in \mathcal{K}_{\mathcal{J}}(m,n)$. Thus $\mathcal{N}_{\mathcal{J}}$ is both invariant and $\mathcal{K}_{\mathcal{J}}$-saturated. Since $\mathcal{O}(\mathcal{N}_{\mathcal{J}}) \subset P$ the identity map factors through to a surjection

$$\Psi : \mathcal{DR}(\mathcal{K}_{\mathcal{J}})/\mathcal{O}(\mathcal{N}_{\mathcal{J}}) \longrightarrow \mathcal{DR}(\mathcal{K}_{\mathcal{J}})/P.$$

As the ideals $\mathcal{O}(\mathcal{N}_{\mathcal{J}})$ and P are gauge invariant the gauge action on $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})$ factors through to gauge actions on $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})/\mathcal{O}(\mathcal{N}_{\mathcal{J}})$ and $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})/P$. Epimorphism Ψ intertwines these actions. In view of Theorem 6.7 we may identify $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})/\mathcal{O}(\mathcal{N}_{\mathcal{J}})$ with $\mathcal{DR}(\mathcal{K}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}})$ and then we have $\Psi(i_{(n,m)}\Big(a + \mathcal{N}_{\mathcal{J}}(m,n))\Big) = i_{(n,m)}(a) + P$, for $a \in \mathcal{K}_{\mathcal{J}}(m.n)$. Thus we see that Ψ injective on every space $i_{(n,m)}\Big(\mathcal{K}_{\mathcal{J}}(m,n) + \mathcal{N}_{\mathcal{J}}(m,n)\Big)$. Consequently Ψ is injective on spectral subspaces of $\mathcal{DR}(\mathcal{K}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}}) = \mathcal{DR}(\mathcal{K}_{\mathcal{J}})/\mathcal{O}(\mathcal{N}_{\mathcal{J}})$, and since it is gauge invariant, it is an isomorphism. Hence

$\mathcal{O}(\mathcal{N}_{\mathcal{J}}) = P$. This together with Theorem 6.7 proves the correspondence between the objects in ii) and iii). ■

Corollary 7.7 (Lattice structure of gauge invariant ideals in $\mathcal{DR}(\mathcal{T})$)**.** *There is a lattice isomorphism* $\mathrm{Lat}_{\mathcal{T}}(\mathcal{T}) \cong \mathrm{Lat}(\mathcal{DR}(\mathcal{T}))$ *between the gauge invariant ideals in* $\mathcal{DR}(\mathcal{T})$ *and invariant* $\mathcal{T}$*-saturated ideals in* $\mathcal{T}$*. Moreover an ideal in* $\mathrm{Lat}(\mathcal{DR}(\mathcal{T}))$ *corresponding to an ideal* $\mathcal{N}$ *in* $\mathrm{Lat}_{\mathcal{T}}(\mathcal{T})$ *is isomorphic to* $\mathcal{DR}(\mathcal{N})$*.*

It follows from Propositions 6.2, 7.1 that for every invariant and $\mathcal{K}_{\mathcal{J}}$-saturated ideal $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$ there is an invariant and $\mathcal{J}$-saturated ideal $\mathcal{N}$ in $\mathcal{T}$ such that $\mathcal{K}\cap\mathcal{N}_{\mathcal{J}} = \mathcal{K}\cap\mathcal{N}$ (under our identification $\mathcal{K}\subset\mathcal{K}_{\mathcal{J}}$ given by (39)). Conversely, any ideal in $\mathcal{K}$ of the form $\mathcal{K}\cap\mathcal{N}$ where $\mathcal{N}$ is invariant and $\mathcal{J}$-saturated ideal in $\mathcal{T}$ give rise to the invariant and $\mathcal{K}_{\mathcal{J}}$-saturated ideal $\mathcal{N}_{\mathcal{J}}[\mathcal{K}\cap\mathcal{N}]$ in $\mathcal{K}_{\mathcal{J}}$ where

$$\mathcal{N}_{\mathcal{J}}[\mathcal{K}\cap\mathcal{N}](r,r+k) := \left\{ \sum_{\substack{j=0,\\ j+k\geq 0}}^{r} i^{\mathcal{M}}_{(j,j+k)}(a_{j,j+k}) : a_{j,j+k} \in (\mathcal{K}\cap\mathcal{N})(j,j+k) \right\}.$$

For an invariant and $\mathcal{K}_{\mathcal{J}}$-saturated ideal $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$ we have $\mathcal{N}_{\mathcal{J}}[\mathcal{K}\cap\mathcal{N}_{\mathcal{J}}]\subset\mathcal{N}_{\mathcal{J}}$ but in general $\mathcal{N}_{\mathcal{J}}[\mathcal{K}\cap\mathcal{N}_{\mathcal{J}}]\neq\mathcal{N}_{\mathcal{J}}$. That is the reason why the embedding $\mathrm{Lat}_{\mathcal{J}}(\mathcal{K}) \hookrightarrow \mathrm{Lat}_{\mathcal{K}_{\mathcal{J}}}(\mathcal{K}_{\mathcal{J}}) \cong \mathrm{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}))$ in general fails to be an isomorphism. We now introduce conditions under which the aforementioned obstacle vanish.

Theorem 7.8. *If one of the conditions hold*

i) $\mathcal{K}\subset\mathcal{J}+\ker\otimes 1$,

ii) $\mathcal{K}$ *admits a transfer action and* $\mathcal{J} = J(\mathcal{K})\cap(\ker(\otimes 1))^{\perp}$,

then every ideal in $\mathrm{Lat}_{\mathcal{K}_{\mathcal{J}}}(\mathcal{K}_{\mathcal{J}})$ *have the form* $\mathcal{N}_{\mathcal{J}}[\mathcal{K}\cap\mathcal{N}]$ *where* $\mathcal{K}\cap\mathcal{N}\in\mathrm{Lat}_{\mathcal{J}}(\mathcal{K})$*. As a consequence we get the lattice isomorphisms*

$$\mathrm{Lat}_{\mathcal{J}}(\mathcal{K}) \cong \mathrm{Lat}_{\mathcal{K}_{\mathcal{J}}}(\mathcal{K}_{\mathcal{J}}) \cong \mathrm{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})).$$

Moreover, every gauge invariant ideal P *in* $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ *is generated by the image of an ideal* $\mathcal{K}\cap\mathcal{N}$ *where* $\mathcal{N}$ *is invariant ideal in* $\mathcal{T}$ *and* $P\cong\mathcal{O}_{\mathcal{T}}(\mathcal{K}\cap\mathcal{N},\mathcal{J}\cap\mathcal{N})$*.*

Proof. Let $\mathcal{N}_{\mathcal{J}}\in\mathrm{Lat}_{\mathcal{K}_{\mathcal{J}}}(\mathcal{K}_{\mathcal{J}})$ and let $\mathcal{N}$ be an invariant and $\mathcal{J}$-saturated ideal in $\mathcal{T}$ such that $\mathcal{K}\cap\mathcal{N}_{\mathcal{J}} = \mathcal{K}\cap\mathcal{N}$. By Proposition 2.6 (and definition of $\mathcal{K}_{\mathcal{J}}$) it suffices to show that an element a in $\mathcal{N}_{\mathcal{J}}(n,n)$ represented by $\sum_{j=0}^{n} i^{\mathcal{M}}_{(j,j)}(a_j)$ where $a_j\in\mathcal{K}(j,j)$, may also be represented by $\sum_{j=0}^{n} i^{\mathcal{M}}_{(j,j)}(b_j)$ where $b_j\in(\mathcal{K}\cap\mathcal{N})(j,j)$, $j=0,..,n$. For abbreviation we shall write equality between a and its representatives.

i). Suppose that $\mathcal{K}\subset\mathcal{J}+\ker\otimes 1$. Then $a_0 = b_0 + j_0$ where $b_0\in(\ker\otimes 1)(0,0)$ and $j\in\mathcal{J}(0,0)$. Plainly $a = i^{\mathcal{M}}_{(0,0)}(b_0) + i^{\mathcal{M}}_{(1,1)}(j_0\otimes 1 + a_1) + \sum_{j=2}^{n} i^{\mathcal{M}}_{(j,j)}(a_j)$. Since $j_0\otimes 1 + a_1$ is in $\mathcal{K}\subset\mathcal{J}+\ker\otimes 1$ we have $j_0\otimes 1+a_1 = b_1+j_1$ where $b_1\in(\ker\otimes 1)(1,1)$, $j_1\in\mathcal{J}(1,1)$ and then $a = i^{\mathcal{M}}_{(0,0)}(b_0) + i^{\mathcal{M}}_{(1,1)}(b_1) + i^{\mathcal{M}}_{(1,1)}(j_1\otimes 1 + a_2) + \sum_{j=3}^{n} i^{\mathcal{M}}_{(j,j)}(a_j)$. Proceeding in this way one gets

$$a = \sum_{j=0}^{n} i^{\mathcal{M}}_{(j,j)}(b_j) \quad \text{where} \quad b_j\in(\ker\otimes 1)(j,j),\ j=0,...,n-1,\ b_n\in\mathcal{K}(n,n).$$

We denote by $\{\mu^{(j)}_{\lambda}\}_{\lambda}$ an approximate unit in $\mathcal{K}(j,j)$, $j=0,...,n$. Using the form of multiplication in $\mathcal{K}_{\mathcal{J}}$ we obtain

$$i^{\mathcal{M}}_{(n,n)}(b_n) = \lim_{\lambda}\ i^{\mathcal{M}}_{(n,n)}(b_n)\star i^{\mathcal{M}}_{(n,n)}(\mu^{(n)}_{\lambda}) = \lim_{\lambda}\ a\star i^{\mathcal{M}}_{(n,n)}(\mu^{(n)}_{\lambda}) \in \mathcal{N}_{\mathcal{J}}(n,n),$$

that is $b_n \in (\mathcal{K}\cap\mathcal{N})(n,n)$. Thus, similar computations show that

$$i^{\mathcal{M}}_{(n-1,n-1)}(b_{n-1}) = \lim_{\lambda} \left(a \star i^{\mathcal{M}}_{(n-1,n-1)}(\mu_\lambda^{(n-1)}) - i^{\mathcal{M}}_{(n,n)}(b_n(\mu_\lambda^{(n-1)} \otimes 1))\right) \in \mathcal{N}_{\mathcal{J}}(n,n).$$

Hence, since $(\otimes_{\mathcal{J}} 1)^{-1}(\mathcal{N}_{\mathcal{J}}) \subset \mathcal{N}_{\mathcal{J}}$, we get $i^{\mathcal{M}}_{(n-1,n-1)}(b_{n-1}) \in \mathcal{N}_{\mathcal{J}}(n-1,n-1)$, that is $b_{n-1} \in (\mathcal{K}\cap\mathcal{N})(n-1,n-1)$. Proceeding in this way one obtains $b_j \in (\mathcal{K}\cap\mathcal{N})(j,j)$, $j=0,..,n$.

ii). Suppose that $\mathcal{J} = J(\mathcal{K}) \cap (\ker(\otimes 1))^{\perp}$ and $\mathcal{K}$ admits a transfer action $\mathcal{L}$. One sees that

$$a = i^{\mathcal{M}}_{(0,0)}(b_0) \quad \text{where} \quad b_0 = \sum_{j=0}^{n} \mathcal{L}^{n-j}(a_j) \in \mathcal{K}(0,0).$$

Since $(\otimes_{\mathcal{J}} 1)^{-1}(\mathcal{N}_{\mathcal{J}}) \subset \mathcal{N}_{\mathcal{J}}$ one gets $i^{\mathcal{M}}_{(0,0)}(b_0) \in \mathcal{N}_{\mathcal{J}}(0,0)$, that is $b_0 \in (\mathcal{K}\cap\mathcal{N})(0,0)$ and the proof is complete. ■

Corollary 7.9. *If $\mathcal{K} \subset \mathcal{J} + \ker \otimes 1$, then a right tensor representation $\pi = \{\pi_{n,m}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ coisometric on $\mathcal{J}$ integrates to the faithful representation of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ if and only if π is gauge invariant, $\mathcal{J}$ is the ideal of coisometricity for π and $\ker\pi = \mathcal{R}_{\mathcal{J}}$.*

Corollary 7.10. *If $\mathcal{K}$ admits a transfer action and $\mathcal{J} = J(\mathcal{K}) \cap (\ker(\otimes 1))^{\perp}$, then a right tensor representation $\pi = \{\pi_{n,m}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ coisometric on $\mathcal{J}$ integrates to the faithful representation of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ if and only if π is faithful and gauge invariant.*

7.1. **Ideal structure of relative Cuntz-Pimsner algebras.** Now we show how a complete description of the gauge invariant ideal structure of relative Cuntz-Pimsner algebras $\mathcal{O}(J,X)$ can be deduced from Theorems 7.6, 7.8. Hopefully it will shed more light on the results of [18]. We start with a useful statement which follows from Theorem 7.8 and contains, as particular cases, [18, Cor. 8.7, Thm. 10.6], [29, Thm. 6.4].

Theorem 7.11. *Let X be a C^*-correspondence over A and let J be an ideal in $J(X)$. If one of the conditions hold*

i) *$A = J + \ker\phi$,*
ii) *X is a Hilbert bimodule and $J = J(X) \cap (\ker\phi)^{\perp}$,*

then we have an isomorphism between the lattice of gauge invariant ideals in $\mathcal{O}(J,X)$ and lattice of X-invariant J-saturated ideals in A:

$$\mathcal{O}(J,X) \,\triangleright\, P \;\mapsto\; i_{00}^{-1}(P) \,\triangleleft\, A.$$

In particular, every gauge invariant ideal P in $\mathcal{O}(J,X)$ is generated by the image of an X-invariant ideal I in A and then P is isomorphic to $\mathcal{O}_X(I, J\cap I)$ and Morita equivalent to $\mathcal{O}(J\cap I, XI)$. If the condition ii) holds we actually have $P \cong \mathcal{O}(J\cap I, XI)$.

Proof. If $A = J + \ker\phi$, then on the level of the C^*-precategory $\mathcal{T}_X$ we have $\mathcal{K}_X \subset \mathcal{K}_X(J) + \ker\otimes 1$. If X is a Hilbert bimodule and $J = J(X)\cap(\ker\phi)^{\perp}$, then by Proposition 5.8, $\mathcal{K}_X$ admits a transfer action and $\mathcal{K}_X(J) = J(\mathcal{K}_X) \cap (\ker\otimes 1)^{\perp}$. Moreover, for every X-invariant J-saturated ideal I in A we have $\phi(I)X = XI$, see [18, Prop. 10.2]. Thus it suffices to apply Theorems 7.8, 6.18. ■

A general result (without additional assumptions) requires description of invariant and $\mathcal{K}_{\mathcal{J}}$-saturated ideals in the right tensor C^*-precategory $\mathcal{K}_{\mathcal{J}}$ where

$$\mathcal{K} := \mathcal{K}_X = \{\mathcal{K}(X^{\otimes n}, X^{\otimes m})\}_{n,m\in\mathbb{N}} \;\text{ and }\; \mathcal{J} := \mathcal{K}_X(J) = \{\mathcal{K}(X^{\otimes n}, X^{\otimes m}J)\}_{n,m\in\mathbb{N}}.$$

Equivalently, by Theorem 7.6, we need to describe all the kernels of right tensor representations $\widetilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ coisometric on $\mathcal{K}_{\mathcal{J}}$. We recall that each such representation $\widetilde{\pi}$ is uniguely determined by a representation (π,t) of X coisometric on J, see Theorem 3.12 and Proposition 7.1.

Proposition 7.12. *Let (π,t) be a faithful representation of X coisometric on J and let I' be the ideal of coisometricity for (π,t) (then automatically $J\subset I'\subset (\ker\phi)^{\perp}\cap J(X)$). The kernel of the corresponding representation $\widetilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ is uniquely determined by I'. Namely, $\ker\widetilde{\pi}(n,n)$, $n\in\mathbb{N}$, consists of elements that can be represented by $\sum_{k=0}^{n} i^{\mathcal{M}}_{(k,k)}(a_k)$, $a_k\in\mathcal{K}(X^{\otimes k})$, such that*

$$\sum_{k=0}^{j} a_k\otimes 1^{j-k}\in\mathcal{K}(X^{\otimes j}I'),\ j=0,...,n-1,\qquad \sum_{k=0}^{n} a_k\otimes 1^{n-k}=0. \tag{42}$$

Proof. Let $a=\sum_{k=0}^{n} i^{\mathcal{M}}_{(k,k)}(a_k)\in\mathcal{K}_{\mathcal{J}}(n,n)$, $a_k\in\mathcal{K}(X^{\otimes k})$, $k=0,...,n$, $n\in\mathbb{N}$, be such that $\widetilde{\pi}(a)=0$. We consider (π,t) (and thus also $\widetilde{\pi}$) as a representation on a Hilbert space H and take advantage of the family $\{P_m\}_{m\in\mathbb{N}}$ of decreasing projections described on page 20. Within the notation of Theorem 3.14, we get

$$\begin{aligned} 0=\|\widetilde{\pi}(a)\| &= \max\left\{\max_{j=0,1,\dots,n-1}\{\|\widetilde{\pi}(a)(P_j-P_{j+1})\|\},\ \left\|\widetilde{\pi}(a)P_n\right\|\right\}\\ &=\max\left\{\max_{j=0,1,\dots,n-1}\{\|\overline{\pi}_{jj}(\sum_{k=0}^{j}a_k\otimes 1^{j-k})(P_j-P_{j+1})\|\},\ \left\|\overline{\pi}_{nn}(\sum_{k=0}^{n}a_k\otimes 1^{n-i})\right\|\right\}. \end{aligned}$$

Thus

$$\overline{\pi}_{jj}(\sum_{k=0}^{j}a_k\otimes 1^{j-k})(P_j-P_{j+1})=0,\quad j=1,...,n-1,\qquad \overline{\pi}_{nn}(\sum_{k=0}^{n}a_k\otimes 1^{n-k})=0. \tag{43}$$

Since π is faithful, $\{\overline{\pi}_{ij}\}_{i,j\in\mathbb{N}}$ is faithful and (43) imply that $\sum_{k=0}^{n}a_k\otimes 1^{n-k}=0$ and each element $\overline{\pi}_{jj}(\sum_{k=0}^{j}a_k\otimes 1^{j-k})$ is supported on $P_{j+1}H$, $j=0,...,n-1$. In particular, we have $\left(\sum_{k=0}^{n-1}a_k\otimes 1^{n-1-k}\right)\otimes 1=\sum_{k=0}^{n}a_k\otimes 1^{n-k}-a_n\in\mathcal{K}(X^{\otimes n})$ and thus by Lemma 3.24 ii), $\sum_{k=0}^{n-1}a_k\otimes 1^{n-1-k}\in\mathcal{K}(X^{\otimes(n-1)}I')$. Similarly, if we assume that $\sum_{k=0}^{n-m}a_k\otimes 1^{n-m-k}\in\mathcal{K}(X^{\otimes(n-m)}I')$, for certain $m=0,...,n-1$, then

$$\left(\sum_{k=0}^{n-m-1}a_k\otimes 1^{n-m-1-k}\right)\otimes 1=\sum_{k=0}^{n-m}a_k\otimes 1^{n-m-k}-a_{n-m}\in\mathcal{K}(X^{\otimes(n-m-1)})$$

and hence by Lemma 3.24 ii), $\sum_{k=0}^{n-m-1}a_k\otimes 1^{n-m-1-k}\in\mathcal{K}(X^{\otimes(n-m-1)}I')$. Thus by induction relations (42) are satisfied. ■

By passing to quotients we may use the above proposition to get a description of the kernel of $\widetilde{\pi}$ in a general situation. To this end we use the following lemma.

Lemma 7.13. *Let (π,t) be a representation of X and let*

$$I=\ker\pi,\qquad I'=\{a\in A:\pi(a)\in\pi_{11}(\mathcal{K}(X))\}. \tag{44}$$

Then representation (π,t) factors through to the faithful representation of X/XI for which the ideal of coisomtericity is I'/I. In particular the following relations hold

$$I\ \ \textit{is}\ \ X\textit{-invariant},\qquad I\subset I'\subset q^{-1}(J(X/XI)\cap(\ker\phi^{I})^{\perp}). \tag{45}$$

where $q:A\to A/I$ is the quotient map and ϕ^I is the left action on X/XI.

Proof. See [18, Lemma 5.10] and Corollary 3.23. ■

Proposition 7.14. *If (π,t) is a representation of X coisometric on J, then the kernel of the corresponding right tensor representations $\widetilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ is uniquely determined by the ideals*

$$I=\ker\pi \qquad and \qquad I'=\{a\in A:\pi(a)\in\pi_{11}(\mathcal{K}(X))\}.$$

Namely, $\ker\widetilde{\pi}(n,n)$, $n\in\mathbb{N}$, consists of elements that can be represented in a form $\sum_{k=0}^{n} i^{\mathcal{M}}_{(k,k)}(a_k)$, $a_k\in\mathcal{K}(X^{\otimes k})$, $k=0,...,n$, where

$$\sum_{k=0}^{j} a_k\otimes 1^{j-k}\in\mathcal{K}(X^{\otimes j}I'),\ j=0,...,n-1,\ \ \sum_{k=0}^{n} a_k\otimes 1^{n-k}\in\mathcal{K}(X^{\otimes n}I).$$

Proof. Pass from (π,t) to the (faithful) quotient representation $(\dot{\pi},\dot{t})$ of the C^*-correspondence X/XI over A/I. By Proposition 6.11 this corresponds to passing from the right tensor representation $\{\pi_{nm}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ to the (faithful) right tensor representation $\{\dot{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}/\mathcal{T}_X(I)=\mathcal{K}_{X/XI}$. Applying Proposition 7.12 to representations $(\dot{\pi},\dot{t})$, $\{\dot{\pi}_{nm}\}_{n,m\in\mathbb{N}}$ we get the assertion by Lemma 7.13. ■

For our purposes it is reasonable to adapt the notion of coisometricity to [18, Defn. 5.6], that is we adopt the following

Definition 7.15. A pair (I,I') of ideals in A satisfying (45) is called a *T-pair* of X. We shall say that a T-pair (I,I') is *coisometric* on an ideal J in A, if $J\subset I'$. In particular, an O-pair introduced in [18, Defn. 5.21] is simply a T-pair coisometric on $J=(\ker\phi)^{\perp}\cap J(X)$.

Let us note that if (I,I') is a T-pair coisometric on J, then I is automatically J-saturated. Indeed, in view of Lemma 7.13 we have

$$J\subset q^{-1}(J(X/XI)\cap(\ker\phi^I)^{\perp}) \quad\Longrightarrow\quad J\cap\varphi^{-1}(\mathcal{K}(XI))\subset I.$$

Furthermore, T-pairs form a lattice with the natural order induced by inclusion.

Theorem 7.16 (Lattice structure of gauge invariant ideals in $\mathcal{O}(J,X)$, cf. [18])**.** *We have lattice isomorphisms between the following objects*

i) *T-pairs (I,I') of X coisometric on J.*
ii) *invariant and $\mathcal{K}_{\mathcal{J}}$-saturated ideals $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$*
iii) *gauge invariant ideals P in $\mathcal{O}(J,X)$*

The correspondence between the objects in ii) and iii) is given by the equality $\mathcal{O}(\mathcal{N}_{\mathcal{J}})=P$. The correspondence between the objects in i) and ii) is given by the equivalence: $a\in\mathcal{K}_{\mathcal{J}}(n,n)$ is in $\mathcal{N}_{\mathcal{J}}(n,n)$, $n\in\mathbb{N}$, iff it may be represented by $\sum_{k=0}^{n} i_{(k,k)}(a_k)$ where

$$\sum_{k=0}^{j} a_k\otimes 1^{j-k}\in\mathcal{K}(X^{\otimes j}I'),\ j=0,...,n-1,\quad \sum_{k=0}^{n} a_k\otimes 1^{n-k}\in\mathcal{K}(X^{\otimes n}I).$$

Moreover, we have

$$P\cong\mathcal{DR}(\mathcal{N}_{\mathcal{J}}),\qquad \mathcal{O}(J,X)/P\cong\mathcal{O}(I'/I,X/XI).$$

Proof. In view of Theorem 7.6 and Proposition 7.14 the only thing we need to show is that the pair of ideals (I,I') from item i) define (via the described equivalence) invariant and $\mathcal{K}_{\mathcal{J}}$-saturated ideal $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$. We note that by (45) and Corollary 4.18 the universal representation of X/XI in $\mathcal{O}(I'/I,X/XI)$ is faithful. Composing this representation with quotient maps $X\to X/XI$, $A\to A/I$ one gets the representation (π,t) of X such that that relations (44) are satisfied. The pair

(π, t) give rise to representation $\widetilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ whose kernel (by Proposition 7.14) is the desired ideal $\mathcal{N}_{\mathcal{J}}$. Using Theorem 7.3 we get $\mathcal{O}(J,X)/P \cong \mathcal{O}(I'/I, X/XI)$. ∎

Remark 7.17. The ideals we identified in Theorem 6.18 as algebras of the form $\mathcal{O}_X(I, J\cap I)$ are exactly these gauge invariant ideals in $\mathcal{O}(J,X)$ that correspond to T-pairs (I, I') where $I' = I + J$. As we noticed in Theorem 7.11 in many situations all gauge invariant ideals are of this form.

8. Embedding conditions for $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$

We fix a right tensor C^*-precategory $\mathcal{T}$ and exhibit conditions for algebras of type $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ to be embedded into one another via the universal representations. These results have similar motivation as [13, Prop. 3.2], [10, Prop. 6.3] and shall be applied in Section 9 to algebras associated with C^*-correspondences. They also may be viewed as a description of certain gauge invariant subalgebras of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. We start with

Proposition 8.1 (Necessary conditions)**.** *For $j = 1,2$, let $\mathcal{K}_j$ and $\mathcal{J}_j$ be ideals in $\mathcal{T}$ such that $\mathcal{J}_j \subset J(\mathcal{K}_j)$ and $\mathcal{K}_1 \subset \mathcal{K}_2$. Denote by $\{i^{(j)}_{(n,m)}\}_{m,n\in\mathbb{N}}$ the universal representations of $\mathcal{K}_j$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_j, \mathcal{J}_j)$, $j = 1,2$. The natural homomorphism $\Psi : \mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J}_1) \longmapsto \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J}_2)$*

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J}_1) \ni i^{(1)}_{(n,m)}(a) \stackrel{\Psi}{\longmapsto} i^{(2)}_{(n,m)}(a) \in \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J}_2) \tag{46}$$

is well define if and only if $\mathcal{J}_1 \subset \mathcal{J}_2$. Moreover, if Ψ is well defined and injective, then

$$\mathcal{J}_1 = \mathcal{J}_2 \cap J(\mathcal{K}_1), \qquad \mathcal{R}_{\mathcal{J}_2} \cap \mathcal{K}_1 = \mathcal{R}_{\mathcal{J}_1}$$

where $\mathcal{R}_{\mathcal{J}_j}$ is the reduction ideal associated with $\mathcal{J}_j$, $j = 1,2$, see Definition 6.8.

Proof. If $\mathcal{J}_1 \subset \mathcal{J}_2$, then Ψ is well defined by the construction of norm in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_j, \mathcal{J}_j)$, $j = 1,2$, see Proposition 4.13. Conversely, if Ψ is well defined, then for $a \in J(\mathcal{K}_1)(m,n)$ we have

$$i^{(1)}_{(n,m)}(a) = i^{(1)}_{(n+1,m+1)}(a\otimes 1) \implies i^{(2)}_{(n,m)}(a) = i^{(2)}_{(n+1,m+1)}(a\otimes 1),$$

that is $\mathcal{J}_1 \subset \mathcal{J}_2$. In the event Ψ is injective the above implication is an equivalence and thus we have $\mathcal{J}_1 = \mathcal{J}_2 \cap J(\mathcal{K}_1)$. Moreover, representations $\{i^{(1)}_{(n,m)}\}_{m,n\in\mathbb{N}}$ and $\{i^{(2)}_{(n,m)}|_{\mathcal{K}_1}\}_{m,n\in\mathbb{N}} = \{\Psi \circ i^{(1)}_{(n,m)}\}_{m,n\in\mathbb{N}}$ have the same kernels and hence we get $\mathcal{R}_{\mathcal{J}_2} \cap \mathcal{K}_1 = \mathcal{R}_{\mathcal{J}_1}$. ∎

In view of the above statement we may limit our attention to algebras $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J}\cap J(\mathcal{K}_1))$ and $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J})$ where $\mathcal{K}_1 \subset \mathcal{K}_2$ and $\mathcal{J} \subset J(\mathcal{K}_2)$.

Proposition 8.2 (Sufficient conditions on ideals $\mathcal{K}_1$ and $\mathcal{J}$)**.** *Let $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{J}$ be ideals in $\mathcal{T}$ such that $\mathcal{K}_1 \subset \mathcal{K}_2$ and $\mathcal{J} \subset J(\mathcal{K}_2)$. The condition*

$$\mathcal{K}_1 \cap J(\mathcal{J}) \subset J(\mathcal{K}_1) \tag{47}$$

(which holds e.g. whenever $\mathcal{J} \subset \mathcal{K}_1$, $\mathcal{J} \subset \mathcal{K}_1^{\perp}$ or $\mathcal{K}_1$ is invariant) implies that there is a natural embedding

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J}\cap J(\mathcal{K}_1)) \subset \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J}).$$

Proof. Notice that $\mathcal{K}_1 \cap J(\mathcal{J}) \subset J(\mathcal{K}_1)$ is equivalent to $\mathcal{J}\cap\mathcal{K}_1\cap(\otimes 1)^{-1}(\mathcal{J}+\mathcal{K}_1) \subset J(\mathcal{K}_1)$. Indeed, on the one hand we have $\mathcal{K}_1 \cap J(\mathcal{J}) \subset \mathcal{J}\cap\mathcal{K}_1\cap(\otimes 1)^{-1}(\mathcal{J}+\mathcal{K}_1)$,

and on the other if $a \in \mathcal{J} \cap \mathcal{K}_1 \cap (\otimes 1)^{-1}(\mathcal{J} + \mathcal{K}_1)(n,m)$, then either $a \in J(\mathcal{K}_1)$ or $a \in \mathcal{K}_1 \cap J(\mathcal{J})$:

$$a \in (\mathcal{K}_1 \cap \mathcal{J})(n,m) \wedge a \otimes 1 \in (\mathcal{K}_1 + \mathcal{J})(n+1,m+1) \implies a \in J(\mathcal{K}_1)(n+1,m+1) \tag{48}$$

where $m, n \in \mathbb{N}$. We show that Ψ given by (46) is faithful on the core (and hence on all of the spectral subspaces) of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1))$. To this end let $a \in \mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1))$ be of the form $a = \sum_{j\in\mathbb{N}} i_{(j,j)}(a_j)$, $a_j \in \mathcal{K}_1(j,j)$, $j \in \mathbb{N}$, and suppose that $\|\Psi(a)\| = 0$. Then by Proposition 4.16 we have

$$\sum_{j=0}^{s} a_j \otimes 1^{s-j} \in \mathcal{J}(s,s), \quad s \in \mathbb{N}, \qquad \lim_{r\to\infty} \sum_{j=0}^{r} a_j \otimes 1^{r-j} = 0.$$

Since $a_0 \in (\mathcal{K}_1 \cap \mathcal{J})(0,0)$ and $a_0 \otimes 1 = (a_0 \otimes 1 + a_1) - a_1 \in (\mathcal{K}_1 + \mathcal{J})(1,1)$, by (48) we get $a_0 \in J(\mathcal{K}_1)(1,1)$ and consequently $a_0 \otimes 1 + a_1 \in (\mathcal{J} \cap \mathcal{K}_1)(1,1)$. Suppose now that $\sum_{j=0}^{s} a_j \otimes 1^{s-j} \in (\mathcal{J} \cap \mathcal{K}_1)(s,s)$ for certain $s \in \mathbb{N}$. Since

$$\left(\sum_{j=0}^{s} a_j \otimes 1^{s-j}\right) \otimes 1 = \sum_{j=0}^{s+1} a_j \otimes 1^{s+1-j} - a_{s+1} \in (\mathcal{J} + \mathcal{K})(s+1, s+1),$$

by (48) we then have $\sum_{j=0}^{s} a_j \otimes 1^{s-j} \in J(\mathcal{K}_1)(s+1, s+1)$ and consequently $\sum_{j=0}^{s+1} a_j \otimes 1^{s+1-j} \in (\mathcal{J} \cap \mathcal{K}_1)(s+1, s+1)$. Thus by induction we get

$$\sum_{j=0}^{s} a_j \otimes 1^{s-j} \in (\mathcal{J} \cap J(\mathcal{K}_1))(s,s), \quad s \in \mathbb{N}, \qquad \lim_{r\to\infty} \sum_{j=0}^{r} a_j \otimes 1^{r-j} = 0$$

which in view of Proposition 4.16 is equivalent to $\|a\| = 0$.
Clearly, $\Psi : \mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1)) \to \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J})$ preserves the gauge actions and hence injectivity of Ψ on spectral subspaces implies the injectivity of Ψ onto the whole algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1))$. ■

Corollary 8.3. *For any ideals $J \subset J(X)$ and I in A such that*

$$I \cap J(XJ) \subset J(XI) \tag{49}$$

(which holds e.g. whenever $J \subset I$, $J \subset I^\perp$ or I is X-invariant) we have the natural embedding

$$\mathcal{O}_X(I, J \cap J(XI)) \subset \mathcal{O}(J, X).$$

Proof. Apply Proposition 8.2 to $\mathcal{K}_2 = \mathcal{K}_X$, $\mathcal{K}_1 = \mathcal{K}_X(I)$ and $\mathcal{J} = \mathcal{K}_X(J)$. □

We notice that if I is X-invariant, then for $J \subset J(X)$ we have $J \cap J(XI) = J \cap I$ and thus the foregoing statement implies the inclusion $\mathcal{O}_X(I, J \cap I) \subset \mathcal{O}(J, X)$ from Theorem 6.18. As the next example shows the condition (47), or more precisely its special case (49), is essential.

Example 8.4. Suppose $A = A_0 \oplus A_0$ where A_0 is a unital C^*-algebra. Consider the ideals $J = A_0 \oplus A_0$, $I = A_0 \oplus \{0\}$ and the C^*-correspondence $X = X_\alpha$ associated with an endomorphism $\alpha : A \to A$ given by the formula $\alpha(a,b) = (0,a)$. Then we have

$$I \cap J(XJ) = A_0 \oplus \{0\} \not\subseteq J(XI) = \{0\} \oplus A_0,$$

so the inclusion (49) fails. On the other hand the algebra $\mathcal{O}_X(I, J \cap J(XI))$ can not be embedded into $C^*(J, X)$ as we have $C^*(J, X) = \{0\}$ and $\mathcal{O}_X(I, J \cap J(XI)) = I$ (the latter relation may be checked using, for instance, Proposition 4.16).

We shall now exploit the role of the condition $\mathcal{R}_{\mathcal{J}} \cap \mathcal{K} = \mathcal{R}_{\mathcal{J}\cap J(\mathcal{K}_1)}$ that appears in Proposition 8.1. One readily checks that $\mathcal{J} \subset \mathcal{K}_1$ implies $\mathcal{R}_{\mathcal{J}} = \mathcal{R}_{\mathcal{J}\cap J(\mathcal{K}_1)}$ (it follows from Propositions 8.1, 8.2). Moreover, we have

$$\mathcal{J} \subset (\ker \otimes 1)^{\perp} \implies \mathcal{R}_{\mathcal{J}} = \mathcal{R}_{\mathcal{J}\cap J(\mathcal{K}_1)} = \{0\}.$$

Thus for C^*-algebras generalizing Katsura's algebras of C^*-correspondences (cf. Remark 4.4) the condition $\mathcal{R}_{\mathcal{J}} = \mathcal{R}_{\mathcal{J}\cap J(\mathcal{K}_1)}$ is trivially satisfied.

Proposition 8.5 (Sufficient conditions on ideal $\mathcal{K}_1$)**.** *Let $\mathcal{K}_1$ be an ideal in $\mathcal{T}$ such that*

$$\otimes^{-1}(\mathcal{K}_1) \cap (\ker \otimes 1)^{\perp} \subset \mathcal{K}_1. \tag{50}$$

Then for every ideal $\mathcal{K}_2$ in $\mathcal{T}$ and every ideal $\mathcal{J}$ in $J(\mathcal{K}_2)$ the natural homomorphism (46) *establish the embedding*

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1)) \subset \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J})$$

if and only if $\mathcal{R}_{\mathcal{J}} \cap \mathcal{K}_1 = \mathcal{R}_{\mathcal{J}\cap J(\mathcal{K}_1)}$.

Proof. The "only if" part follows from Proposition 8.1. Let us assume that $\mathcal{R}_{\mathcal{J}} \cap \mathcal{K}_1 = \mathcal{R}_{\mathcal{J}\cap J(\mathcal{K}_1)}$. By the reduction procedure described in Theorem 6.9 applied to $\mathcal{J}$ we may actually assume that $\mathcal{R}_{\mathcal{J}} = \{0\}$ and $\mathcal{J} \subset (\ker \otimes 1)^{\perp}$. It suffices to prove that if $a \in \mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1))$ is such that $a = \sum_{s=0}^{r} i_{(s,s)}(a_s)$, $a_s \in \mathcal{K}_1(s,s)$, $s = 0, ..., r$, and $\|\Psi(a)\| = 0$, then $\|a\| = 0$. In view of Proposition 4.16 the requirement $\|\Psi(a)\| = 0$ is equivalent to

$$\sum_{j=0}^{s} a_j \otimes 1^{s-j} \in \mathcal{J}(s,s), \quad s = 0, ..., r-1, \qquad \sum_{j=0}^{r} a_j \otimes 1^{r-j} = 0. \tag{51}$$

To show that $\|a\| = 0$ we need to check if

$$\sum_{j=0}^{s} a_j \otimes 1^{s-j} \in (\mathcal{J} \cap J(\mathcal{K}_1))(s,s), \quad s = 0, ..., r-1, \qquad \sum_{j=0}^{r} a_j \otimes 1^{r-j} = 0. \tag{52}$$

However, since

$$\Big(\sum_{j=0}^{r-1} a_j \otimes 1^{r-1-j}\Big) \otimes 1 = \sum_{j=0}^{r} a_j \otimes 1^{r-j} - a_r = -a_r \in \mathcal{K}_1(r,r),$$

it follows that $\sum_{j=0}^{r-1} a_j \otimes 1^{r-1-j} \in \Big(\otimes^{-1}(\mathcal{K}_1) \cap (\ker \otimes 1)^{\perp}\Big)(r-1, r-1) \subset J(\mathcal{K}_1)(r-1, r-1)$. Analogously, if $\sum_{j=0}^{s} a_j \otimes 1^{s-j} \in J(\mathcal{K}_1)(s,s)$, $s = 0, ..., r-1$, then

$$\Big(\sum_{j=0}^{s-1} a_j \otimes 1^{s-1-j}\Big) \otimes 1 = \sum_{j=0}^{s} a_j \otimes 1^{s-j} - a_{s,s} \in \mathcal{K}_1(r,r)$$

and $\sum_{j=0}^{s-1} a_j \otimes 1^{s-1-j} \in \Big(\otimes^{-1}(\mathcal{K}_1) \cap (\ker \otimes 1)^{\perp}\Big)(s-1, s-1) \subset J(\mathcal{K}_1)(s-1, s-1)$. Hence by induction (52) holds. ■

Item ii) of Lemma 1.8 could be stated as that the ideal $\mathcal{K}_X = \{\mathcal{K}(X^{\otimes n}, X^{\otimes m})\}_{m,n\in\mathbb{N}}$ in $\mathcal{T}_X$ satisfies (50), and thus we get

Proposition 8.6. *Let $\mathcal{T} = \mathcal{T}_X$ be a right tensor C^*-precategory of a C^*-correspondence X and let $\mathcal{S}$ and $\mathcal{J}$ be ideals in $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K}_2)$ and $\mathcal{K}_X \subset \mathcal{K}_2$. If we put $J := \mathcal{J}(0,0)$, then the natural homomorphism is an embedding*

$$\mathcal{O}(J \cap J(X), X) \subset \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J}),$$

if and only if $R_J = R_{J\cap J(X)}$ where R_J and $R_{J\cap J(X)}$ are reducing ideals associated to J and $J\cap J(X)$ respectively, see Definition 6.20.

In view of Proposition 5.1, the algebra $\mathcal{O}_{\mathcal{T}_X}(\mathcal{T}_X,\mathcal{T}_X)$ coincides with a *Doplicher-Roberts algebra of a C^*-correspondence* investigated in [10], [13]. It is natural to consider the following "relative version" of such algebras.

Definition 8.7. Suppose X is a C^*-correspondence over A and J is an arbitrary ideal in A. We shall call the C^*-algebra

$$\mathcal{DR}(J,X) := \mathcal{O}_{\mathcal{T}_X}(\mathcal{T}_X,\mathcal{T}_X(J))$$

a *relative Doplicher-Roberts algebra of X* relative to J. Within this notation the algebra considered in [10], [13] is $\mathcal{DR}(A,X)$.

Now Proposition 8.6 can be interpreted as the following generalization of [10, Prop. 6.3], [13, Prop. 3.2].

Corollary 8.8. *The natural homomorphism is an embedding*

$$\mathcal{O}(J\cap J(X),X)\subset \mathcal{DR}(J,X).$$

if and only if $R_J = R_{J\cap J(X)}$. In particular, $\mathcal{O}(J\cap J(X),X)$ embeds into $\mathcal{DR}(J,X)$ whenever $J\subset (\ker\phi)^\perp$ or $J\subset J(X)$.

Example 8.9. Let A_0 be a non-unital C^*-algebra and A_0^+ its minimal unitization. Let us consider the C^*-correspondence $X = A_0\oplus A_0^+$ over $A = A_0^+\oplus A_0^+$ where $\langle x,y\rangle_A := x^*y$, $x\cdot a = xa$ and $a\cdot x = \alpha(a)x$ where $\alpha(a_0\oplus b_0) = 0\oplus a_0$. Then $J(X) = A_0^+\oplus A_0^+$ and for the ideal $J=A$ we have

$$R_J = A_0^+\oplus A_0^+ \neq A_0\oplus A_0^+ = R_{J\cap J(X)}.$$

On the other hand $\mathcal{O}(J\cap J(X),X)\cong\mathbb{C}$ and $\mathcal{DR}(J,X)=\{0\}$.

9. Application to Doplicher-Roberts algebras associated with C^*-correspondences

In this section our study is motivated by [10], [13] and our aim is to generalize [10, thm. 6.6], [13, Thm. 4.1]. We recall that there is a one-to-one correspondence between representations Ψ of the algebra $\mathcal{O}(J,X)$ and representations (π,t) of X coisometric on J, which is given by relations

$$\pi = \Psi\circ i_{(0,0)}|_A,\qquad t=\Psi\circ i_{(1,0)}|_X.$$

The inverse of this correspondence is given by the equality $\Psi=\Psi_{[\pi,t]}$ where $[\pi,t]$ is the right tensor representation of $\mathcal{K}_X$ defined in Proposition 3.12. For shortening we shall denote the representation of $\mathcal{O}(J,X)$ corresponding to (π,t) by $\pi\times_J t$, cf. [10]. We arrive at the following statement, cf. [18, Cor. 11.7], [29, thm. 5.1], [17, 6.4].

Theorem 9.1 (Gauge invariance theorem for $\mathcal{O}(J,X)$)**.** *Let (π,t) be a representation of a C^*-correspondence X coisometric on an ideal $J\subset J(X)$. Then $\pi\times_J t$ is a faithful representation of $\mathcal{O}(J,X)$ if and only if (π,t) admits a gauge action,* $\ker\pi = R_J$ *and*

$$J=\{a\in A: \pi(a)\in\pi_{11}(\mathcal{K}(X))\}. \tag{53}$$

In particular,

i) *if $J\subset(\ker\varphi)^\perp$, then $\pi\times_J t$ is faithful if and only if (π,t) is faithful, admits a gauge action and J is ideal of coisometricity for (π,t).*

ii) *if* $J+(\ker\varphi)^{\perp}=A$, *then* $\pi\times_J t$ *is faithful if and only if* (π,t) *admits a gauge action,* $\ker\pi=R_J$ *and* J *is ideal of coisometricity for* (π,t).

Proof. For the first part use Theorem 7.3 and Proposition 7.14. To see items i), ii) apply Proposition 7.12 and Theorem 7.11. ■

Remark 9.2. One could state the foregoing theorem in a somewhat more general form similar to Theorem 7.3. Namely, if in the above statement one drops the requirement of admitting a gauge action, one gets necessary and sufficient conditions for $\pi\times_J t$ to be faithful on spectral subspaces of $\mathcal{O}(J,X)$.

If (π,t) is a representation on a Hilbert space H, then $[\pi,t]=\{\pi_{nm}\}_{m,n\in\mathbb{N}}$ extends to a right tensor representation $\overline{[\pi,t]}=\{\overline{\pi}_{nm}\}_{m,n\in\mathbb{N}}$ of $\mathcal{T}_X$, see Proposition 3.8. Thus, for appropriately chosen ideal J, $\overline{[\pi,t]}$ integrates to a representation of relative Doplicher-Roberts algebra $\mathcal{DR}(J,X)$ that we shall denote by $\overline{\pi\times_J t}$. In particular by Theorem 3.14 we have

Proposition 9.3. *Let* J *be an ideal in* A. *A representation* Ψ *of* $\mathcal{DR}(J,X)$ *on a Hilbert space* H *is of the form* $\overline{t\times_J\pi}$ *for a representation* (π,t) *of* X *if and only if*

$$\Psi(i_{n,n}(\mathcal{L}(X^{\otimes n}))H=\Psi(i_{n,n}(\mathcal{K}(X^{\otimes n}))H,\qquad n\in\mathbb{N},$$

where $\{i_{(n,m)}\}_{m,n\in\mathbb{N}}$ *is the universal representations of* $\mathcal{T}_X$ *in* $\mathcal{DR}(J,X)$. *If this is the case and additionally* $R_J=R_{J_0}$ *where* $J_0=J\cap J(X)$, *then* $\overline{t\times_J\pi}$ *is an extension of* $t\times_{J_0}\pi$ *(as we then have* $\mathcal{O}(J_0,X)\subset\mathcal{DR}(J,X)$, *cf. Corollary 8.8).*

We note that

$$\{a\in A:\overline{\pi}_{11}(\phi(a))=\pi(a)\}=\{a\in A:\pi(a)P_1=\pi(a)\}$$
$$=\{a\in A:\pi(a)\in\overline{\pi}_{11}(\mathcal{L}(X))\}=\{a\in A:\pi(a)H\subset\overline{t(X)H}\},$$

cf. Lemma 3.24 i) or [10, Lem. 1.9], so the forthcoming results may be stated without a use of the mapping $\overline{\pi}_{11}$. We shall however keep using it as it indicates the relationship with coisometricity for C^*-correspondences. The following statement generalizes [10, Thm. 6.6].

Theorem 9.4 (Extensions of representations from $\mathcal{O}(J_0,X)$ up to $\mathcal{DR}(J,X)$)**.** *Let* (π,t) *be a representation of* X *on a Hilbert space* H *and let* J *be an ideal in* A *such that*

$$J\subset\{a\in A:\overline{\pi}_{11}(\phi(a))=\pi(a)\},\quad \text{and}\quad R_J=R_{J_0}\quad\text{where}\quad J_0=J\cap J(X).$$

Then $\mathcal{O}(J_0,X)\subset\mathcal{DR}(J,X)$ *and the representation* $\overline{\pi\times_J t}$ *of* $\mathcal{DR}(J,X)$ *is an extension of the representation* $\pi\times_{J_0}t$ *of* $\mathcal{O}(J_0,X)$ *and*

i) $\overline{\pi\times_J t}$ *is faithful on spectral subspaces of* $\mathcal{DR}(J,X)$ *if and only if* $J=\{a\in A:\overline{\pi}_{11}(\phi(a))=\pi(a)\}$ *and* $\pi\times_{J_0}t$ *is faithful on spectral subspaces of* $\mathcal{O}(J_0,X)$.

ii) $\overline{\pi\times_J t}$ *is faithful if and only if* $J=\{a\in A:\overline{\pi}_{11}(\phi(a))=\pi(a)\}$ *and* $\pi\times_{J_0}t$ *is faithful.*

Proof. i). Since $\pi\times_{J_0}t$ is considered as a restriction of $\overline{\pi\times_J t}$, faithfulness of $\overline{\pi\times_J t}$ on spectral subspaces, implies such a faithfulness of $\pi\times_{J_0}t$, and then we get $J=\{a\in A:\overline{\pi}_{11}(\phi(a))=\pi(a)\}$ by Proposition 4.19. Conversely, suppose that $J=\{a\in A:\overline{\pi}_{11}(\phi(a))=\pi(a)\}$ and $\pi\times_{J_0}t$ is faithful on spectral subspaces of $\mathcal{O}(J_0,X)$. In this event $\ker\pi=R_J=R_{J_0}$ and hence by passing to quotients (dividing all the associated C^*-precategories by $\mathcal{N}=\mathcal{T}_X(\ker\pi)$) we may assume that $\ker\pi=0$. Then, by Theorem 3.14, representation $\{\overline{\pi}_{ij}\}_{i,j\in\mathbb{N}}$ is faithful. To

show that $\overline{\pi \times_J t}$ is faithful let $a = \sum_{s=0}^{r} i_{(s,s)}(a_s)$, where $a_s \in \mathcal{L}(X^{\otimes s})$, $s = 0, ..., r$, $r \in \mathbb{N}$, be such that $\overline{\pi \times_J t}(a) = 0$. Similarly as in the proof of Proposition 7.12 we get

$$\overline{\pi}_{ss}(\sum_{j=0}^{s} a_j \otimes 1^{s-j})(P_s - P_{s+1}) = 0, \quad s = 1, ..., r-1, \qquad \overline{\pi}_{rr}(\sum_{j=0}^{r} a_j \otimes 1^{r-j}) = 0.$$

In particular, $\sum_{j=0}^{r} a_j \otimes 1^{r-j} = 0$ and each operator $\overline{\pi}_{ss}(\sum_{j=0}^{s} a_j \otimes 1^{s-j})$ is supported on $P_{s+1}H$, $s = 0, ..., r-1$. Thus applying inductively Lemma 3.24 i) we get

$$\sum_{j=0}^{s} a_j \otimes 1^{s-j} \in \mathcal{L}(X^{\otimes s}, X^{\otimes s}J), \quad s = 1, ..., r-1, \qquad \sum_{j=0}^{r} a_j \otimes 1^{r-j} = 0,$$

which is equivalent to $\|a\| = 0$.
ii). If $\overline{\pi \times_J t}$ is faithful, then $\pi \times_{J_0} t$ is faithful and $J = \{a \in A : \overline{\pi}_{11}(\phi(a)) = \pi(a)\}$ by Proposition 4.19. Conversely, if $J = \{a \in A : \overline{\pi}_{11}(\phi(a)) = \pi(a)\}$ and $\pi \times_{J_0} t$ is faithful, then $\overline{\pi \times_J t}$ is faithful on spectral subspaces of $\mathcal{DR}(J, X)$ by item i). Thus it suffices to show that

$$\|\overline{\pi \times_J t}(E(a))\| \leq \|\overline{\pi \times_J t}(a)\|, \qquad a \in \mathcal{DR}(J, X), \tag{54}$$

where E is conditional expectation onto 0-spectral subspace of $\mathcal{DR}(J, X)$, see [12, Thm 4.2]. For that purpose we note three facts. Firstly, we have

$$\|\overline{\pi \times_J t}(E(a))\| \leq \|\overline{\pi \times_J t}(a)\|, \qquad a \in \mathcal{O}(J_0, X) \subset \mathcal{DR}(J, X), \tag{55}$$

(a wicker version of (54)) because $\overline{\pi \times_J t}$ is an extension of $\pi \times_{J_0} t$ and $\pi \times_{J_0} t$ is faithful. Secondly, for any $a \in \mathcal{DR}(J, X)$ and any $r \in \mathbb{N}$ we have

$$\|\overline{\pi \times_J t}(a)\| = \max\left\{ \max_{s=0,1,...,r-1} \left\{ \|\overline{\pi \times_J t}(a)(P_s - P_{s+1})\| \right\}, \left\|\overline{\pi \times_J t}(a)P_r\right\| \right\}, \tag{56}$$

where $\{P_m\}_{m \in \mathbb{N}}$ is the family of decreasing projections described on page 20. Thirdly, it is enough to prove (54) for elements $a \in \mathcal{DR}(J, X)$ of the form

$$a = \sum_{k=-\infty}^{\infty} \sum_{\substack{j=0, \\ j+k \geq 0}}^{r} i_{(j+k,j)}(a_{j+k,j}), \qquad a_{j+k,j} \in \mathcal{L}(X^{\otimes j}, X^{\otimes j+k}), \ r \in \mathbb{N}, \tag{57}$$

as they form a dense subspace of $\mathcal{DR}(J, X)$.
Thus we fix an element a of the form (57), $\varepsilon > 0$ and $\xi \in (P_s - P_{s+1})H$ where $s = 0, 1, ..., r-1$. Since π_{ss} restricted to $P_s H$ is nondegenerate, by Hewitt-Cohen Factorization Theorem there exist $c \in \mathcal{K}(X^{\otimes s})$ and $\eta \in H$ such that $\xi = \pi_{ss}(c)\eta$ and

$$\|\pi_{ss}(c)\| \cdot \|\eta\| \leq (1 + \varepsilon) \cdot \|\xi\|.$$

We have

$$\begin{aligned}\|\overline{\pi\times_J t}(E(a))\xi\| &= \|\overline{\pi\times_J t}(E(a))(P_s-P_{s+1})\xi\| = \|\overline{\pi}_{ss}\big(\sum_{j=0}^{s} a_{j,j}\otimes 1^{s-j}\big)\xi\|\\ &= \|\overline{\pi}_{ss}\big(\sum_{j=0}^{s} a_{j,j}\otimes 1^{s-j}\big)\pi_{ss}(c)\eta\| = \|\pi_{ss}\Big(\big(\sum_{j=0}^{s} a_{j,j}\otimes 1^{s-j}\big)c\Big)\eta\|\\ &\le \|\overline{\pi\times_J t}\Big(i_{(s,s)}\big(\sum_{j=0}^{s} a_{j,j}\otimes 1^{s-j}c\big)\Big)\|\cdot\|\eta\|\\ &= \|\overline{\pi\times_J t}\Big(E\big(\sum_{k=-\infty}^{+\infty} i_{(s+k,s)}\big(\sum_{\substack{j=0,\\ j+k\ge 0}}^{s} a_{j+k,j}\otimes 1^{s-j}c\big)\big)\Big)\|\cdot\|\eta\|.\end{aligned}$$

Applying (55) to $\sum_{k=-\infty}^{+\infty} i_{(s+k,s)}(\sum_{\substack{j=0,\\ j+k\ge 0}}^{s} a_{j+k,j}\otimes 1^{s-j}c)\in\mathcal{O}(J_0,X)$ we get

$$\begin{aligned}\|\overline{\pi\times_J t}(E(a))\xi\| &\le \|\overline{\pi\times_J t}\Big(\sum_{k=-\infty}^{+\infty} i_{(s+k,s)}\big(\sum_{\substack{j=0,\\ j+k\ge 0}}^{s} a_{j+k,j}\otimes 1^{s-j}c\big)\Big)\|\cdot\|\eta\|\\ &\le \|\sum_{k=-\infty}^{+\infty}\overline{\pi}_{s+k,s}\big(\sum_{\substack{j=0,\\ j+k\ge 0}}^{s} a_{j+k,j}\otimes 1^{s-j}\big)\|\cdot\|\pi_{ss}(c)\|\cdot\|\eta\|\\ &\le \|\sum_{k=-\infty}^{+\infty}\sum_{\substack{j=0,\\ j+k\ge 0}}^{r}\overline{\pi}_{j+k,j}(a_{j+k,j})(P_s-P_{s+1})\|\cdot(1+\varepsilon)\cdot\|\xi\|\\ &= \|\overline{\pi\times_J t}(a)(P_s-P_{s+1})\|\cdot(1+\varepsilon)\cdot\|\xi\|.\end{aligned}$$

Therefore by arbitrariness of ε and ξ we have

$$\|\overline{\pi\times_J t}(E(a))(P_s-P_{s+1})\|\le\|\overline{\pi\times_J t}(a)(P_s-P_{s+1})\|\quad\text{for}\quad s=0,...,r-1.$$

Similarly one shows that $\|\overline{\pi\times_J t}(E(a))P_r\|\le\|\overline{\pi\times_J t}(a)P_r\|$. Hence by (56) inequality (54) holds and the proof is complete.
■

In the above theorem we required that $R_J=R_{J_0}$. This condition however is automatically satisfied whenever one deals with faithful representations. In particular every faithful representation of $\mathcal{O}(J_0,X)$ extends to a faithful representation of $\mathcal{DR}(J,X)$ for an appropriate ideal J. Indeed, we have

Theorem 9.5. *Let (π,t) be a representation of X on Hilbert space H. Put*

$$J=\{a\in A:\overline{\pi}_{11}(\phi(a))=\pi(a)\}\qquad\text{and}\qquad J_0=J\cap J(X).$$

If $\pi\times_{J_0} t$ is faithful on $\mathcal{O}(J_0,X)$ (resp. on spectral subspaces of $\mathcal{O}(J_0,X)$), then $R_J=R_{J_0}$ and representation $\overline{\pi\times_J t}$ is faithful on $\mathcal{DR}(J,X)$ (resp. on spectral subspaces of $\mathcal{DR}(J,X)$).

Proof. Representations $\pi\times_{J_0} t$ and $\overline{\pi\times_J t}$ are intertwined by the natural homomorphism Ψ, see Proposition 8.1:

$$(\pi\times_{J_0} t)(a)=\overline{\pi\times_J t}\ (\Psi(a)),\qquad a\in\mathcal{O}(J_0,X).$$

Hence, if $\pi\times_{J_0} t$ is faithful on spectral subspaces of $\mathcal{O}(J_0,X)$, then so is Ψ. As faithfulness of Ψ on spectral subspaces of $\mathcal{O}(J_0,X)$ implies the equality $R_J=R_{J_0}$ the assertion follows from Theorem 9.4. ■

Corollary 9.6. *If $\pi \times_{J_0} t$ is a faithful representation of $\mathcal{O}(J_0, X)$ on a Hilbert space H, then*

$$J_0 = J \cap J(X) \quad \text{and} \quad R_J = R_{J_0} \quad \text{where} \quad J = \{a \in A : \overline{\pi}_{11}(\phi(a)) = \pi(a)\},$$

and $\pi \times_{J_0} t$ extends to a faithful representation $\overline{\pi \times_J t}$ of $\mathcal{DR}(J, X)$.

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